[illegible]

NAPIER of MERCHISTON

"*The famous inventor of the Logarithms, the Person to whom the title of a* **Great Man** *is more justly due, than to any other, whom his country ever produced.*" ——— Hume's Hist. vol. vii. p. 85. 8th edit. 1778.

Engraved from a drawing by Mr Brown in the possession of the Earl of Buchan

AN

ACCOUNT

OF THE

LIFE, WRITINGS, and INVENTIONS

OF

JOHN NAPIER,

OF

MERCHISTON;

BY

DAVID STEWART, EARL OF BUCHAN,

AND

WALTER MINTO, L. L. D.

ILLUSTRATED WITH COPPERPLATES.

QUANDO ULLUM INVENIES PAREM?

PERTH:

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M,DCC,LXXXVII.

TO THE

K I N G.

SIR,

AS the writings of Archimedes were addreſſed to the King of Sicily, who had peruſed and reliſhed them, ſo I do myſelf the honour, to addreſs to Your Majeſty, the following account of the Life, Writings, and Inventions of our Britiſh Archimedes, in which, I can claim no other merit, than having endeavoured to call forth and illuſtrate the abilities of others. I feel great pleaſure, in dedicating this Tract to Your Majeſty, after the chaſte and dignified model of Antiquity, beſtowing on the King, the merited encomium, of having promoted the Sciences and Arts, with which it is connected; and in aſſuring Your Majeſty, that I am, with the greateſt reſpect,

Your MAJESTY'S

Moſt dutiful Subject, and

Obedient humble Servant,

BUCHAN.

ADVERTISEMENT.

ABOUT twenty years ago, I thought it would be easy to bring together a groupe of learned men, who would dedicate a part of their leisure to erect literary monuments to the memory of their illustrious countrymen, whose lives had not been hitherto written or sufficiently illustrated; and I wished such monuments to be fashioned and executed by men personally eminent in the departments which distinguished the subjects of their biographical research, and not by the assistants of a bookseller or compiler, who cannot be expected, however faithful and accurate, to be animated with that love to the subject, which the Italian Artist rightly considers as the soul of his enterprize, and the source of its perfection.

In this expectation I have been disappointed; and though I allow the highest merit to the British Biography, now republishing by Dr Kippis, yet in the immense extent of such an undertaking, I perceive the impossibility of its reaching the perfection I have proposed, without the addition of supplementary articles and connections, which would have been in a great measure unnecessary, had my plan been adopted; because the articles, being written with care and with zeal, so as to support themselves in an isolated state by the public favour, would afterwards have been taken up by subsequent editions into that great repository of biographical learning, in a highly finished state, and purged of the errors which are unavoidable, in the first fabric of works of that nature.

With respect to the biography of Scotland, one of the judges there, who would have done it honour in its best days, by his virtue, his attention to the dignity and duties of his station, and the useful employment of his leisure, has generously offered, by an advertisement annexed to the Annals of the Lives of John Barclay, Author of Argenis, and some other learned Scots, to forward the undertaking I wish to promote.

Encouraged by the assistance of an associate, so able and so liberal, I have presumed to offer the following Biographical Tract to the public, as my mite to a Treasury, which I hope to see enriched by many, who have the ability and the generosity of my respectable coadjutor. It was indeed by that excellent man, that I was originally encouraged to prosecute researches of this nature. He applauded that disposition in a young man of quality, which leads him to the study of the history of his own country, not in pamphlets, satires, apologies and panegyrics, but in the private undisguised correspondence of the great.

A man who studies history in this way, will see that the same characters are represented by different actors: introduced behind the scenes, he will see folly dressing itself in the garb of wisdom, and selfishness assuming the mask of public spirit; and among the learned, the plagiary stealing away the laurels of the modest inventor. He will see great events arising from inconsiderable causes, and men neither devils nor angels, but a composition of good and bad qualities, such as the men of the world can see them every day in common life.

I flatter myself, that this article of Napier, in the Biographia Scotica, will be considered in some respects, as a specimen of the plan I have described, for it certainly has been written con amore. *In the scientific part I have received the assistance of a gentleman, who deserves to be better known, on account of his mathematical learning, and the accuracy with which he treats the subjects of his inquiry.*

If

If the following publication, ſhall have the good fortune to meet with the approbation of the learned world, 'tis my intention, to give an account of the lives and writings of Andrew Fletcher of Salton, and John Law of Lauricſton, on the ſame plan. The firſt undertaking will furniſh me an opportunity, of repreſenting the ancient conſtitution of Scotland, in what I apprehend to be a clearer light, than has hitherto been offered; and of treating the cauſes and conſequences of the union between the two kingdoms: and the other will open an ample field for exhibiting the diſorders in the finances of France, occaſioned by the expenſive wars of Lewis the fourteenth, and the Miſſiſippi Scheme, and for explaining by what means they have been gradually remedied and brought to a ſtate, which has enabled that nation, not only to bring her naval force and her trade to a dangerous rivalſhip with this country, but to obtain that credit, by good faith, which in former times, had given ſo decided a ſuperiority to Britain. I am very ſenſible that there are many men in this country much better qualified for performing theſe taſks than I am, and I think it an honour to enjoy their friendſhip: but men of great reputation generally ſeek for reſt in the evening of life, and avoid expoſing their laurels to the blaſt of envy, in their declining years.

Theſe, I hope, will be accepted as ſufficient apologies, for my venturing to occupy ſuch ground, and I beg leave to invite my learned countrymen, to aid me in ſo noble an undertaking, as that of raiſing monuments to the memory of the illuſtrious dead.

I have only to add, that if the ſeparate lives of illuſtrious perſons, ſhould be written on the plan I propoſe, and were accompanied by portraits, elegantly engraven by the beſt artiſts, and the whole executed in a ſimilar manner, of the ſame Quarto ſize, and with the ſame Type and Paper, they would gradually form the nobleſt work, which has been offered to the republic of letters, in any age or country.

AN

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ACCOUNT

OF THE

LIFE, WRITINGS, and INVENTIONS,

OF

JOHN NAPIER,

OF

MERCHISTON.

I Have undertaken to write the Life of JOHN NAPIER, of Merchiſton, a man famous all the world over, for his great and fortunate diſcovery of Logarithms in Trigonometry, by which the eaſe and expedition in calculation, have ſo wonderfully aſſiſted the Science of Aſtronomy, and the arts of practical Geometry and Navigation.

ELEVATED above the age in which he lived, and a benefactor to the world in general, he deſerves the epithet of *Great.*

NAPIER lived in a country of proud Barons, where barbarous hoſpitality, hunting, the military art, and religious controverſy, occupied

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the time and attention of his contemporaries, and where he had no learned ſociety to aſſiſt him in his reſearches.

THIS extraordinary perſon was born at Merchiſton, in the neighbourhood of Edinburgh, in the year 1550*.

HE was the Son of Sir Archibald Napier, of Merchiſton, Maſter of the Mint in Scotland, and of Janet Bothwell, daughter of Mr Francis Bothwell, one of the Senators of the college of Juſtice †.

THAT his family was of ancient eſtabliſhment in the counties of Dunbarton and Stirling, appears from the public records, and from the private archieves of his houſe.

JOHN de Napier, from whom he ſprung in the 12th generation, was one of thoſe proprietors of lands, who ſwore allegiance to Edward the firſt, of England, in the year 1296. William, from whom he counted in the ninth generation, was Governor of the Caſtle of Edinburgh, in the year 1401, whoſe ſon Alexander, was the firſt Baron or Laird of Merchiſton, and was the Father of another of the ſame name, who was Vice Admiral of Scotland, and one of the Commiſſioners from king James the third, at the court of London, in the years 1461 and 1464.

FROM the family of Lennox, Earl of Lennox, he derived a coheirſhip by the marriage of Elizabeth Mentieth, of Ruſky, to his great-grandfather's

* As appears by an inſcription on his portrait, engraved by old Cooper, from an original painting.

† Craufurd's Peerage.

father's father, Sir John Napier, of Merchiston: but on his anceftors he reflected more honour and celebrity than he received, and his name will probably be famous, when the lineage of Plantagenet will be remembred only by genealogifts, and when pofterity may know no more of his, than we now know of the families of Plato, Ariftotle, Archimedes, or Euclid.

It is fit, that men fhould be taught to aim at higher and more permanent glory than wealth, office, titles or parade can afford; and I like the tafk, of making fuch great men look little, by comparing them with men who refemble the fubject of my prefent enquiry.

From Napier's own authority, we learn, that he was educated at St. Andrews *, where writes he, " in my tender years and bairn-age, at " fchools, having on the one part contracted a loving familiaritie with " a certain gentleman a papift, and on the other part being attentive to " the fermons of that worthy man of God, Maifter Chriftopher Good-" man, teaching upon the Apocalyps, I was moved in admiration againft " the blindnefs of papifts that could not moft evidentlie fee their feven " hilled Citie of Rome, painted out there fo lively by Saint John, as the " Mother of all Spiritual Whoredome: that not onlie burfted I oute in " continuall reafoning againft my faid familiar, but alfo from thence-" forth I determined with myfelf by the affiftance of God's fpirit to " employ my ftudy and diligence to fearch out the remanent myfteries " of that holy booke (as to this houre praifed be the Lord I have bin " doing at all fuch times as convenientlie I might have occafion) &c.

THE

* Preface to his plain difcovery of the Revelation of St. John.

The time of Napier's matriculation does not appear from the Regiſter of the Univerſity of St. Andrews, as the books aſcend no higher than the beginning of the laſt century; but as the old whore of Babylon, aſſumed in the eyes of the people of Scotland, her deepeſt tinge of ſcarlet about the year 1566, and as that time correſponds to the literary bairn-age of John Napier, I ſuppoſe, he then imbibed the holy fears and commentaries of Maiſter Chriſtopher Goodman, and as other great Mathematicians have ended, ſo he began his career with that myſterious book.

I have not been able to trace Merchiſton from the Univerſity, till the publication of his Plain Diſcovery, at Edinburgh, in the year 1593 *; though Mackenzie in his lives and characters of the moſt eminent writers of the Scotiſh nation, informs us (without quotation, however, of any authority) that he paſſed ſome years abroad, in the low countries, France and Italy, and that he applied himſelf there, to the ſtudy of Mathematics.

In the Britiſh Muſeum there are two copies of his letter to Anthony Bacon, the original of which, is in the Archbiſhop's Library at Lambeth, entitled " Secret Inventions, profitable and neceſſary, in theſe days, for the defence of this Iſland, and withſtanding ſtrangers enemies to God's truth and religion," which I have cauſed to be printed, in the Appendix to this Tract. This letter is dated, June 7, 1596 †, about which time it appears, as ſhall be ſhewn hereafter, that he had ſet himſelf to explore his Logarithmic Canon.

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* Printed by Andrew Hart, 4to. † Ayſcongles lat. v. 1. p. 155. See Appendix N°, 1.

I have enquired, without fuccefs, among all the defcendants of this eminent perfon, for papers or letters, which might elucidate this dark part of his hiftory; and if we confider that Napier was a reclufe mathematician, living in a country, very inacceffible to literary correfpondence, we have not much room to expect, that the moft diligent explorations, would furnifh much to the purpofe, of having the progrefs of his ftudies.

Among Mr Briggs's papers, preferved in the Britifh Mufeum, I looked for letters from Napier, but found only what Mr Briggs calls, his *Imitatio Nepeirea, five applicatio omnium fere regularum, fuis Logarithmis pertinentium, ad Logarithmos;* which feems to have been written in the year 1614, foon after the publication of the Canon*.

Though the life of a learned man is commonly barren of events, and beft unfolded in the account of his writings, difcoveries, improvements, and correfpondence with the learned men of his age, yet I anxioufly fought for fomewhat more, with refpect to a character, I fo much admired; but my refearches have hitherto been fruitlefs. Perhaps from the letters, books, and collections of focieties or of learned individuals, to which I have not had accefs, fomething may hereafter be brought to light: and one of the inducements, to offer a fketch of this kind to the public, is the tendency it may have to bring forth fuch information. His plain difcovery has been printed abroad, in feveral languages, particularly in French, at Rochelle, in the year 1603, 8vo. an-

D nounced

* Ayfcough's Cat. vol. 1. p. 389.

nounced in the title, as revised by himself *. Nothing could be more agreeable to the Rochellers, or to the Hugonots of France, at this time, than the Author's annunciation of the Pope, as Antichrist, which in this book he has endeavoured to set forth, with much zeal and erudition.

That Napier had begun, about the year 1593, that train of enquiry, which led him to his great atchievement in Arithmetic, appears from a letter to Crugerus from Kepler, in the year 1624; wherein, mentioning the Canon Mirificus, he writes thus, *Nihil autem supra Neperianam rationem esse puto: etsi Scotus quidam literis ad Tychonem, anno* 1594, *Scriptis jam spem fecit Canonis illius mirifici*, which allusion agrees with the idle story mentioned by Wood in his Athenæ Oxon, and explains it in a way perfectly consonant to the rights of Napier as the inventor; concerning which, I shall take occasion to comment, in the account of his works: nor is it to be supposed, that had this noble discovery been properly applied to science, by Justus Byrgius, or Longomontanus, Napier would have been universally acknowledged by his contemporaries, as the undisputed Author of it.

No men in the world, are so jealous of each other as the learned, and the least plausible pretence of this sort, could not have failed to produce

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* This edition was published on the first day of that year, in the end of which the Synod of Gap did declare, or moved to declare, the Pope to be Antichrist, which had never been before attempted, by any body of Protestants. See Sully's Memoirs.

With respect to Napier's fanciful calculation of the completion of the prophecies, concerning the duration of the world, the year, in which this monument is erected to his memory, immediately succeeds that fixed for the end of the world, and no doubt must be the year of judgment, with respect to the authenticity of his discovery, and the merit of those arguments, which are brought forward to support his claim.

a controverſy, in the republic of letters, both in his lifetime, and after his death, when his praiſes were ſounded all over Europe *.

WHEN

* To quote authorities in this place, would be to give a catalogue of all the Mathematical and Arithmetical books of that age.

His moſt outrageous panegyriſt, is Sir Thomas Urquhart, of Cromarty, who has given us alſo ſo ridiculous an account of the admirable Crichton.

In his Jewel, Urquhart, after having referred his readers to his Trigonometrical Work, entitled Triſſotetras, for the praiſes of Napier, thus mentions " an almoſt incomprehenſible device, which be-" ing in the mouths of the moſt of Scotland, and yet unknown to any that ever was in the world but " himſelf, deſerveth very well to be taken notice of in this place; and it is this: he had the ſkill, as " is commonly reported, to frame an engine, (for invention not much unlike that of Archyteas's Dove) " which by virtue of ſome ſecret ſprings, inward reſſorts, with other implements, and materials fit for " the purpoſe, incloſed within the bowels thereof, had the power (if proportionable in bulk to the " action required of it (for he could have made it of all ſizes) to clear a field of four miles circum-" ference, of all the living creatures exceeding a foot in heighth, that ſhould be found thereon, how " near ſoever they might be found to one another; by which means he made it appear, that he was " able, with the help of this machine alone, to kill thirty thouſand Turks, without the hazard of one " Chriſtian!" Of this it is ſaid that (on a wager) he gave proof upon a large plain in Scotland, to the deſtruction of a great many head of Cattle, and flocks of ſheep, whereof ſome were diſtant from other half a mile on all ſides, and ſome a whole mile. To continue the thread of my ſtory, as I have it, I muſt not forget, that when he was moſt earneſtly deſired by an old acquaintance, and profeſſed friend of his, even about the time of his contracting the diſeaſe whereof he died, that he would be pleaſed, for the honour of his family, and his own everlaſting memory to poſterity, to reveal unto him the manner of the contrivance of ſo ingenious a myſtery, ſubjoining thereto, for the better perſuading him, that it were a thouſand pities, that ſo excellent an Invention ſhould be buried with him in the grave, and that after his deceaſe nothing ſhould be known thereof: his anſwer was, that for the ruin and overthrow of man, there were too many devices already framed, which if he could make to be fewer, he would with all his might endeavour to do; and that, therefore, ſeeing the malice and rancor rooted in the heart of mankind, will not ſuffer them to diminiſh the number of them, by any new concert of his ſhould never be increaſed. Divinely ſpoken truly.

Urquhart's Tracts, Edinburgh, 1774. 8vo. p. 57.

When Napier had communicated to Mr Henry Briggs, Mathematical Profeſſor in Greſham College, his wonderful Canon for the Logarithms, that learned Profeſſor ſet himſelf to apply the rules in his *Imitatio Nepeirea*, which I have already mentioned, and in a letter to Archbiſhop Uſher, in the year 1615, he writes thus, " Napier, Lord of " Merchiſton, hath ſet my head and hands at work with his new and " admireable Logarithms. I hope to ſee him this ſummer if it pleaſe " God, for I never ſaw a book which pleaſed me better, and made me " more wonder"*.

It may ſeem extraordinary to quote Lilly the aſtrologer with reſpect to ſo great a man as Napier; yet as the paſſage I propoſe to tranſcribe from Lilly's life, gives a pictureſque view of the meeting betwixt Briggs and the Inventor of the Logarithms, at Merchiſton near Edinburgh, I ſhall ſet it down in the original words of that mountebank knave †.

" I will acquaint you with one memorable ſtory, related unto me by John Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was ſervant to king James the firſt and Charles the firſt. When Merchiſton firſt publiſhed his Logarithms, Mr Briggs then reader of the Aſtronomy Lectures at Greſham College in London, was ſo ſurpriſed with admiration of them, that he could have no quietneſs in himſelf, until he had ſeen that noble perſon whoſe only invention they were: He acquaints John Marr therewith, who went into Scotland before Mr Briggs, purpoſely to be there when theſe two ſo learned perſons ſhould meet; Mr Briggs appoints a certain day when

to

* Uſher's Letters, p. 36. † Lilly's Life, London, 1721. 8vo.

to meet at Edinburgh, but failing thereof, Merchiſton was fearful he would not come. It happened one day as John Marr and the Lord Napier were ſpeaking of Mr Briggs; "Ah John, ſaith Merchiſton, Mr Briggs will not now come": at the very inſtant one knocks at the gate; John Marr haſted down and it proved to be Mr Briggs to his great contentment. He brings Mr Briggs up into My Lord's chamber, where almoſt one quarter of an hour was ſpent, each beholding other with admiration before one word was ſpoken: at laſt Mr Briggs began. "My "Lord I have undertaken this long journey purpoſely to ſee your per- "ſon, and to know by what engine of wit or ingenuity you came firſt "to think of this moſt excellent help unto Aſtronomy, *viz.* the Loga- "rithms; but My Lord, being by you found out, I wonder nobody "elſe found it out before, when now being known it appears ſo eaſy". He was nobly entertained by the Lord Napier, and every ſummer after that during the Laird's being alive, this venerable man Mr Briggs went purpoſely to Scotland to viſit him."

THERE is a paſſage in the life of Tycho Brahe by Gaſſendi*, which may miſlead an inattentive reader to ſuppoſe that Napier's method had been explored by Herwart at Hoenburg, 'tis in Gaſſendi's obſervations on a letter from Tycho to Herwart, of the laſt day of Auguſt 1599. *Dixit Hervartus nihil morari ſe ſolvendi cujuſquem trianguli difficultatem; ſolere ſe enim multiplicationum, ac diviſionum vice additiones ſolum, ſubtractiones*
93 *uſurpare (quod ut fieri poſſet, docuit poſtmodum ſuo Logarithmorum Canone Neperus.)* But Herwart here alludes to his work afterwards publiſhed

* Tychonis Brahæi Vita. Pariſius 4to, 1654. p. 191.

in the year 1610, which ſolves triangles by Proſthaphæreſis, a mode totally different from that of the Logarithms.

KEPLER dedicated his Ephemerides to Napier, which were publiſhed in the year 1617 *; and it appears from many paſſages in his letter about this time, that he held Napier to be the greateſt man of his age, in the particular department to which he applied his abilities: and indeed if we conſider, that Napier's diſcovery was not, like thoſe of Kepler or of Newton, connected with any analogies or coincidences, which might have led him to it, but the fruit of unaſſiſted reaſon and ſcience, we ſhall be vindicated in placing him in one of the higheſt niches in the Temple of Fame.

KEPLER had made many unſucceſsful attempts to diſcover his canon for the periodic motions of the planets and hit upon it at laſt, as he himſelf candidly owns, on the 15th of May, 1618; and Newton applied the palpable tendency of heavy bodies to the earth to the ſyſtem of the univerſe in general; but Napier ſought out his admirable rules, by a ſlow ſcientific progreſs, ariſing from the gradual revolution of truth.

THE laſt literary exertion of this eminent perſon, was the publication of his Rabdology and Promptuary, in the year 1617, which he dedicated to the Chancellor Seton, and ſoon after died at Merchiſton, on the 3d of April, O. S. of the ſame year, in the 68th year of his age, and, as I ſuppoſe, in the 23d of his happy invention.

IN-

* Kepler's Ephemerides novæ motuum cæleſtium ab anno 1617.

In his perfon, the portraits * I have feen reprefent him of a grave and fweet countenance, not unlike his eminent contemporary Monfieur de Peirefc.

In his family he feems to have been uncommonly fortunate, for his eldeft fon became learned and eminent even in his father's lifetime, his third a pupil of his own in Mathematics, to whom he left the care of publifhing his Pofthumous works; and lofing none of his children by death, he loft all his daughters by honourable or refpectable marriages.

He was twice married. By his firft wife, Margaret, the daughter of Sir James Stirling of Kier, defcended of one of the oldeft and moft refpectable gentlemen's families in Scotland, he had an only child, Archibald, his fucceffor in his eftates, of whom I fhall hereafter give fome account. By his fecond marriage with Agnes, the daughter of Sir James Chifholm, of Crombie, he had five fons: John, Laird of Eafter Tonie; Robert †, who publifhed his father's works, whom I have already mentioned, the anceftor of the Napiers of Kilkroigh in Stirling fhire; Alexander Napier of Gillets, Efq; William Napier of Ardmore; and Adam, of whom the Napiers of Blackftone and Craigannet in Stirling fhire are defcended. His daughters were, Margaret, the wife of Sir James Stuart of Roffayth; Jane, married to James Hamilton, Laird of Kilbrachmont in Fife; Elizabeth, to William Cuninghame of Craigends; Agnes, to George Drummond of Baloch; and Helen, to The Reverend

* In the Univerfity Library at Edinburgh, another in the poffeffion of the Lord Napier, and an 8vo print, engraved by Delaram, where he is reprefented calculating with his bones.

† Robert wrote a Chemical Treatife, ftill preferved in the family of Napier.

Reverend Mr Mathew Busbane, Rector of the Parish of Erskine in Renfrew shire.

He was interred in the Cathedral Church of St Giles, at Edinburgh, on the east side of its northern entrance, where there is now a Stone Tablet, indicating, by a Latin Inscription, that the burial place of the Napiers, is in that place; but no Tomb has ever been erected to the memory of so celebrated a man, nor can it be required to preserve his memory, since the astronomer, geographer, navigator and political arithmetician, must feel themselves every day indebted to his inventions, and thus a monument is erected to the illustrious Napier, which cannot be obliterated by time, or depretiated by the ingenuity of others in the same department.

I proceed now to evince more fully the merit of Napier, by giving an account of the state in which he found Arithmetic, and of the benefit it received from his discoveries.

SECTION

SECTION I.

CONCERNING ARITHMETIC.

An eum Statuas et Imagines, non animorum ſimulacra ſed corporum, ſtudioſe multi ſummi homines reliquerunt; conſiliorum relinquere, ac virtutum noſtrarum effigiem nonne multo malle debemus, ſummis ingeniis expreſſam et politam?

CICERONIS ORATIO PRO ARCHIA POETA. CAP. XII.

ARITHMETIC is ſo neceſſary to man, that it muſt have made its appearance on the firſt and rudeſt ſtage of ſociety.

SIGNS to expreſs numbers were probably in uſe, as ſoon as ſigns to expreſs other ideas.

THE ſigns the moſt obvious, and we may venture to ſay the firſt in uſe, were the fingers. The number of theſe accounts for the general adoption of numeration by tens. The firſt ten numbers have the appellation of digits or fingers, in moſt of the languages.

THE next improvement of Arithmetic, ſeems to have been the uſe of ſmall pebbles, or of knotted ſtrings. The words λιθάριον and *calculus* ſignify both a pebble and an arithmetical operation. The Ruſſians, to this day, perform their calculations by means of ſtringed beads, with great exactneſs and expedition. The Greeks and Romans repreſented numbers by the letters of the Alphabet variouſly combined. By means of their notation, the operations of addition and ſubtraction of integers at leaſt, were eaſily enough performed. But multiplication, diviſion, and the extraction of roots, were difficult and tedious operations. They muſt have effected them, in a great meaſure, by dint of thought. Boethius, who flouriſhed towards the end of the fourth century, ſays indeed, that ſome of the Pythagoreans had invented, and uſed in their calculations, nine apices or characters, reſembling thoſe we now employ; by which theſe latter operations muſt have been much ſimplified. Theſe figures were known only to a few myſterious men, and it is by no means probable that they were the inventors of them. It is probable that Pythagoras, or ſome of his diſciples, borrowed them, as they did many other inventions, from the Indians. The merit of the Greek Philoſophers, of which Euclid claims a diſtinguiſhed ſhare, conſiſted in raiſing Arithmetic, from being a ſimple art, to the rank of the ſciences.

Gerbert of Aquitaine, in France, afterwards Pope Sylveſter the ſecond, having imbibed the elements of the ſciences, found that the chriſtian world, at that time involved in darkneſs, could not furniſh him with ſufficient helps for making any great progreſs in them. This induced him to fly from the Convent of Fleury, where he had lived from his infancy, to Spain; where, under the tuition of the Moors, he became ſo intimately acquainted with the mathematics, that he is ſaid to have ſoon ſurpaſſed his maſters. Upon his return to his native coun-

try,

try, about the year 960, or 970, he introduced the ten characters, which form the basis of our modern Arithmetic. These had been familiar to the Arabs, time out of mind, and the invention of them is, by their writers, ascribed to the Indians *.

About five hundred years afterwards, our Arithmetic received a most important improvement, by the invention of decimal fractions.

As the invention of these fractions, and of the Logarithms, with other arithmetical improvements, was occasioned by the efforts of ingenious men, to perfect Trigonometry, it will be proper to give some account of the rise and progress of this most useful branch of the mathematics.

Trigonometry, considered as a simple art, must have begun with the division of lands in every country; but considered as a science, or as the application of Arithmetic to Geometry, it seems to have had its rise among the hands of the great Hipparchus, about one hundred and forty years before the christian æra. Hipparchus was the first who made use of the longitudes and latitudes, for determining the position of places, on the surface of the earth. Theon cites a treatise of his, in twelve books, on the chords of circular arcs, which must have been a treatise on Trigonometry, and is the first of which history gives any account. Menelaus, about the end of the first century, wrote a treatise, in six books, on the chords; and there are extant of his three books on Spherical Trigonometry, where that subject is treated in a manner very profound and extensive.

The

* Wallis, Montucla, &c.

The difficulties to be encountered in the ſolution of triangles, which is the object of Trigonometry, regard the tables of the parts of the circle, the form of the problems to be uſed, and the application of theſe problems to practice.

The Ancients, before Ptolemy's time, do not ſeem to have agreed upon a particular diviſion of the radius of the circle *. That indefatigable Aſtronomer, who flouriſhed about the year 200, having ſimplified the theory of Menelaus, divided the radius into ſixty equal parts, and computed on this foundation, the length of all the chords in the ſemicircle, correſponding to every thirty minutes. This ſexagenary diviſion, which continued in uſe for many centuries, obliged geometers to make uſe of numbers compoſed of integers and fractions, which occaſioned much labour and much loſs of time. The table of chords led them to problems very complicated and of difficult execution. Every oblique triangle was to be divided into two rectangular ones; and in order to come at a ſolution, it was neceſſary to raiſe to the ſquare, and to extract the ſquare root of many fractional numbers.

The Arabs, ſometime in the eleventh century, greatly ſimplified the theory of Trigonometry, by ſubſtituting, for the chords of the double arcs, the halves of theſe chords. Theſe lines have been called ſinus, probably from S. Ins. an abbreviation of the Latin words *ſemiſſes inſcriptarum* †. This improvement paved the way to more ſimple theorems, of which we ſhall have occaſion afterwards to ſpeak.

About

* Hanſchii Pref. in Kepl. Epiſt. † Montucla Hiſtoire Mathematique.

About the middle of the fifteenth century, George, furnamed Peurbach, from a village on the confines of Auftria and Bavaria, where he was born, either adopted the finus from the Arabs, or invented them himfelf. He alfo banifhed from Trigonometry the ufe of the fexagenary calculus, by fuppofing the radius to confift of 600 000 equal parts, and computing on this foundation the length of the fines correfponding to every ten minutes of the Quadrant.

John Muller (commonly known by the name of Regiomontanus from the place of his birth, Konigfberg a town in Franconia) the difciple of Peurbach, improved his mafter's idea by making the radius equal to unity or 1,0000000. On this new plan he calculated, with great labour and accuracy, a table of the fines for all the minutes of the Quadrant. He alfo was the firft who introduced the ufe of the tangents in Trigonometry; of which Erafmus Reinoldus of Salfeldt firft conftructed a table. To thefe tables Rheticus * afterwards added that of the fecants, which had been invented by F. Maurolycus of Meffina.

By means of thefe new tables the art of Trigonometry was not only rendered more accurate than formerly, but one multiplication or divifion was fuperfeded in every geometrical proportion where the radius made one of the terms. The multiplication or divifion, however, of fuch large numbers required much expence of time, labour and attention.

* George Joachim, fo called from his native country. Thefe appellatives, fo much ufed after the revival of letters, make it often difficult to difcover the real names of learned men.

RAYMARUS Urſus, towards the end of the ſixteenth century, having either learned from his preceptor Juſtus Byrgius, or diſcovered ſome new properties of the ſines, ſhewed, in his *Fundamentum Aſtronomicum* publiſhed in the year 1588, how theſe might be employed to great advantage in the ſolution of ſome trigonometrical queſtions. By his method, which he calls *Proſthaphæreſis*, from προσθεσις *additio* and ἀφαίρεσις *ablatio*, the fourth term of a geometrical propoſition, having for its firſt term the radius equal to unity, may be found by addition and ſubtraction only; inſtead, for example, of multiplying the ſine *a* by the ſine of *b* in the geometrical proportion 1 : ſin. *a* : : ſin. b : ſin. *c*, the ſine of *c* may be had, with much leſs trouble, by ſubtracting half the coſine of the ſum of *a* and *b* from half the coſine of their difference; becauſe, as is eaſily demonſtrated, ſin. $a \times$ ſin. $b = \frac{1}{2}$ coſin $(a - b) - \frac{1}{2}$ coſin $(a + b)$.

IT was only to a few caſes, however, that the proſthaphæreſis of Raymar could be applied, and the improvements made upon it, by Clavius * Magini † and others, required ſo many precautions that they were not of very great ſervice. ‡ But inconſiderable as theſe abbreviations of calculus were, they were generally uſed by the moſt eminent mathematicians and aſtronomers at the end of the ſixteenth and beginning of the ſeventeenth century §.

SUCH

* Clavius de Aſtrolabio, book 1. lemma 53.

† Magini primum mobili, lib. 1. theor. 33. and lib. 11. cap. 2.

‡ Quod vero Proſthaphæreſean tabulus attinet, ſcito me totum hunc annum qua parte et a morbis et a curis fui vacuus in unius martis proſthaphæreſibus excentri verſari, nec pudet dicere me ſcopum nondum attigeſſe. Kepler Epiſt. p. 171.

§ Proſthaphæretical tables were publiſhed by J. G. Herwart, in 1610.

Such was the ſtate of arithmetical computation, at the time of the invention of the Logarithms, which, as Napier himſelf ſays, *Omnem illam priſtinæ matheſeos difficultatem penitas e medio tollit; et ad ſublevandam memoriæ imbecillitatem ita ſe accomodat, ut illius adminiculo facile ſit, plures quæſtiones mathematicas unius horæ ſpatio, quam priſtinia et communiter recepta forma ſinuum, tangentium et ſecantium, vel integro die abſolvere**. But before we proceed to this moſt important diſcovery, we ſhall give an account of thoſe ingenious contrivances, intended to anſwer the ſame purpoſe, which previouſly occurred to Napier.

SECTION II.

NAPIER'S BONES.

THE firſt of theſe mechanical devices is what our author calls Rabdologia, or the art of computing by figured rods. Theſe rods are ſquare parallelepipeds three inches in length, and three tenths in breadth. Each of the faces of theſe parallelepipeds is divided into ten equal parts, of which nine are ſquares and in the middle, and half of the tenth at one extremity or the top, and half at the other extremity or the bottom. Every one of theſe ſquares is cut by a diagonal from left to right upwards. At the top of each face is ſome one of the ten digits 0, 1, 2, 3, &c.

In

* Log. Canon. deſcriptio. in dedic.

In the firſt ſquare below that digit is repeated, in the ſecond is its double, in the third it's triple, and ſo on. Of theſe multiples of the digit, the figure of units is below, and the figure of tens above the diagonal. The meaning of what has been juſt ſaid will be evident by a little attention to Fig. I. where the four faces of each rod of the ſet, recommended by Napier, are unfolded. By means of theſe rods the operations of multiplication and diviſion are performed by addition and ſubtraction.

The rule for multiplication is—Bring the rods to form the multiplicand at the top of their upper face. Join a rod, having unity at the top of its upper face, to the right or left hand ſide; in which ſeek the right hand figure of the multiplicator, and write out the numbers correſponding thereto in the ſquare of the other faces, by adding the ſeveral numbers occurring in the ſame rhomboid. Seek the ſecond figure of the multiplicator and proceed in the ſame manner: arrange and add the numbers wrote out, as in common multiplication; the ſum is the product required. To multiply 1785 by 364, for example, I diſpoſe the proper rods as in Fig. II.; next to 4 (the firſt right hand figure of the multiplicator) I find 0; in the contiguous rhomboid 2 and 2, which added together make 4; in the next 3 and 8 which make 1 and a ſurplus of ten; and in the laſt 2 and 4 which, together with unity for the ten I had in the former rhomboid, make 7. Theſe numbers 0, 4, 1, 7, I ſet down one after the other as I find them, proceeding from right to left. I go on in the ſame manner with 6 and 3 (the other figures of the multiplicator); and, after arranging and adding the partial products I find the total product required. Thus,

```
   364
  ----
  7140
 10710
 5355
 ------
 649740
```

The rule for division. BRING the rods to form the divisor at the top of their upper face. Join a rod having unity at the top of its upper face, to the right or left hand side. Descend under the divisor till you meet those figures of the dividend wherein it is first required how often the divisor is found, or the next less number, which subtract from the first figures of the dividend, and put for the first figure of the quotient the corresponding number on the side face. Bring down, one after the other, the remaining figures of your dividend as in common division, and proceed in the same manner till you have finished the operation. Let it, for example, be required to divide 649740 by 364. I dispose the rods as in Fig. III. The next less number under the divisor 364 to 649 (the first figures of the dividend) I find to be 364 itself which I subtract from 649 putting 1, the number corresponding on the side face, for the first figure of my quotient: to my remainder 285 I bring down 7 the next figure of my dividend. The next less number to 2857 under the divisor I find to be 2548, which I subtract from 2857, putting 7, the number corresponding in the side face, for the second figure of the quotient. I go on in the same manner till I have brought down the other figures of the dividend and completed my quotient as follows.

```
649740 (1785
364
----
2857
2548
-----
 3094
 2912
 -----
  1820
  1820
  -----
  (0)
```

Although the extraction of the ſquare and cube roots may be pretty expeditiouſly performed by the rods, Napier propoſes an auxiliary lamella for the abridgement of it. It would ſerve little purpoſe to give a particular deſcription of the lamella, or an account of the manner of uſing it. Its length and thickneſs are the ſame with thoſe of the rods, and its breadth quadruple. Its two faces are divided and marked as in Fig. IV. To find out the way of operating by it will be no difficulty to any body who is a little acquainted with arithmetic and has time to ſpare.

Another of Napier's contrivances is his *multiplicationis promptuarium.*

This machine conſiſts of a box of figured lamellæ. The lamellæ, two hundred in number, are each eleven inches in length and one inch in breadth. Each of theſe lamellæ is divided into eleven equal parts of which ten in the middle are ſquares, and two thirds of the eleventh at one

one extremity, and one third at the other. Every one of theſe ſquares is divided into nine leſs ſquares, one hundred of the lamellæ are each one fourth of an inch in thickneſs, and the other hundred one eighth. Suppoſe the former, which we ſhall call direct lamellæ, placed ſo that the greater margin may appear at top and the leſs at bottom; and the latter which we ſhall call tranſverſe, placed laterally, with the greater margin to the right and the leſs to the left hand. In this poſition every ſquare appears cut by a diagonal (faint in the ſmall but ſtrong in the great ones) from the left to right upwards. Each of every ten both of the direct and of the tranſverſe lamellæ has ſome one of the ten digits 0, 1, 2, 3, *&c.* inſcribed on its greater margin. The multiples of the digit on the margin of a direct lamella are diſpoſed in each of its greater ſquares as pointed out by Fig. V. where *a* repreſents the digit itſelf, *b* the right hand figure, and *b′* the left hand figure of its double; *c* and *c′* the right and left hand figures of its triple (the plain letters being above and the accented ones below the diagonal of the figure); *d* and *d′* thoſe of its quadruple, and ſo on. In the tranſverſe lamellæ thoſe which have *o* on the margin are untouched; thoſe which have unity on the margin have the loculus correſponding to *a* cut out; thoſe which have two on the margin have the loculi of 6 and 6′ cut out; thoſe which have 3 the loculi of *c* and *c′*; thoſe which have 4 the loculi of *d* and *d′*, *&c.* This will be ſufficiently evident by inſpecting Fig. VI. where it is examplified in a direct lamella titled with the digit 4, and in a tranſverſe one with 7. The box fitted to receive theſe lamellæ is of a cubical form; ſomething more than eleven inches wide and nearly eight inches high; ſee Fig. VII. Two of its contiguous ſides, which we ſhall diſtinguiſh by the names of left and right, are divided

divided into twenty parts, each equal in length to the breadth of ten lamellæ, and in height to the thickneſs of a direct and of a tranſverſe lamella alternately. The greater diviſions on the left ſide are cut out, and the leſs on the right ſide. Into the box through each of the former, with their titled ends foremoſt, ten direct lamellæ of the ſame title are inſerted with their untitled ends foremoſt, and an equal number of the tranſverſe ones of the ſame title, through each of the latter. Thoſe titled *o* are in the uppermoſt diviſions, and thoſe titled 1, 2, 3, *&c.* in the reſpective diviſions below.

Multiplication by the promptuary is performed as follows. The firſt, or right hand, ſecond, third, &c. figure of the multiplicand is exhibited by the title of a lamella taken from the firſt, or right hand, ſecond, third, &c. column of the left ſide of the box and placed on its lid exactly above and as it lay in that column. The empty ſpace, if any, towards the left is to be covered with blank lamellæ. The firſt, or right hand, ſecond, third, &c. figure of the multiplicator is exhibited by the title of a lamella taken from the firſt, or left hand, ſecond, third, &c. column of the right ſide of the box and placed on the former lamella exactly above, and as it lay in that column. The remaining ſpaces, if any, towards the right are to be covered with blank lamellæ. All the uſeful multiples on the direct lamellæ appear through the *feneſtellæ*, and all the uſeleſs multiples are hid. All the numbers beginning at the corner next the firſt or right hand figures of the multiplicand and multiplicator, lying between the united ſtrong diagonals, are to be added ſeverally; the right hand figures of the ſums, marked down; and 1 for every 10, carried to the next place, till we come to the oppo-

ſite

ſite corner: and the work is done. This operation, we truſt, is deſcribed with ſufficient accuracy and plainneſs to ſuperſede the neceſſity of an example. In order that diviſion may be performed by the Promptuary, it muſt firſt be converted into multiplication by means of tables dreſſed on purpoſe, or of tables of the ſines, tangents and ſecants, conſtructed on the hypotheſis of the radius being equal to unity, followed by a certain number of Zeros. That this may be accompliſhed by theſe laſt, look for the co-ſecant, or co-tangent of the arc which has the diviſor for its ſine or tangent. Make the co-ſecant or co-tangent found the multiplicand, and the dividend the multiplicator; or converſely. Find the product by the promptuary as above directed. This product, a number of the right hand figures according to the number of zeros in the ſquare of the radius being marked off as decimals, is the quotient required. The reaſon of this is obvious: the co-ſecants or co-tangents being third proportionals to the ſines or tangents and the radius or unity; to multiply any number by one of the two firſt, or to divide it by the correſponding one of the two ſecond of theſe lines, is one and the ſame thing.

LOCAL ARITHMETIC.

LOCAL Arithmetic, another ingenious invention of Napier, is the art of calculating by means of counters properly placed on a chefs board, or fimilar table. Two contiguous margins (which we fhall diftinguifh by the names of left or inferior, and right or lateral) of that table, are divided into a number of parts equal to that of their adjoining fquares. The inferior divifions beginning at the right and the lateral at the left have fucceffively infcribed in them the fucceffive terms of the geometrical progreffion 1, 2, 4, 8, 16, 32, &c. which are called local numbers.

COMMON numbers are reduced to local by fubtraction, and local numbers to common by addition. The common number 1875, for example, expreffed in local numbers will be found to be 1024; 512; 256; 64; 16; 2 and 1: and *vice verfa.* The firft of thefe reductions is neceffary before, and the fecond after any of the operations of common arithmetic are performed by this contrivance. By the help of a very fimple table, reduction may be performed with eafe and expedition.

To Add. PUT a counter for each local number in the proper place on the lateral or on the inferior margin of your table. For every two counters found in the fame place, put one in the next higher, after removing *them.* Repeat this till no place fhall contain more than one counter. The counters left indicate the numbers required. Thus let it be required to find the fum 1875; 258, and 1099. I put the coun-

ters

ters at 1024; 512; 256; 64; 16; 2 and 1, the local numbers of the firſt; at 256, and 2, thoſe of the ſecond; and at 1024; 64, 8, 2, and 1, thoſe of the third. At 1.—I find two counters which I remove, and put a counter at 2 where I find other three. I take up theſe four and put two, in the next place 4 &c. and proceeding in this manner I find at laſt a counter at each of the following numbers 2048; 1024; 128, and 32, which form 3232 the ſum ſought.

To Subtract. PUT a counter for each local number of the greater of the two quantities, at its proper place, a little to one ſide, on the inferior margin; and one for each of the local numbers of the leſs of the two quantities, at its proper place, a little to the other ſide, on the ſame margin. Remove the counters found on oppoſite ſides of the ſame place. Change the ſide of the loweſt counter remaining; take off that above it; and put a counter in each place between them. Remove as before. Repeat this till there ſhall be no counters on one of the ſides of the margin; and thoſe on the other will indicate the remainder. Let it be propoſed, for example, to ſubtract 1099 from 1875. I put counters at 1024; 64; 8; 2 and 1, to the left of the lateral margin, and at 1024; 512; 256; 64; 16; 2 and 1, to the ſit of that margin. Finding a counter on each ſide of the numbers 1024; 64; 2, and 1, I remove them. My loweſt counter is to the left of 8. I put it to the right and take up 16. above it; as there are no intermediate places, and as the remaining counters are on the ſame ſide of the margin my operation is finiſhed. The remainder is 512; 256, and 8; or 776.

MULTIPLICATION,

MULTIPLICATION, Divifion, and the extraction of the fquare root, may alfo be performed on the margin: but they are performed with much greater eafe and clearnefs on two contiguous margins and the fquares of the table. On thefe laft the counters have two different movements; the one parallel to the fides like that of the tour, and the other diagonal like that of the bifhop, on the chefs board. Every fquare of the table is faid to have for its value one of the equal numbers (on the two margins) between which it lies diagonally. The two fides of a fquare formed by counters in the area of the table, parallel to the inferior and lateral margins, we fhall call a Gnomon: this gnomon confifting of 3, 5, 7, &c. counters is faid to be congruous when its value can be fubtracted from the numbers left marked upon the margin.

To Multiply. MARK with counters the local multiplicator in the inferior and the local multiplicator in the lateral margin. From the middle of the marked places let points be fuppofed to move perpendicularly into the table, and put a counter at each interfection. Remove the counters on the margins. Bring the counters in the fquares of the tables to their values in one of the margins; add, if neceffary, and the work is done. Suppofe, for example, 19 is to be multiplied by 13. I mark with counters Fig. VIII. the numbers 1, 2, and 16, on the inferior and the numbers 1, 4, and 8, on the lateral margin, having placed other counters rectangularly in the table, I remove the marginal ones. Thofe other counters I bring up, one by one, to their proper places in the lateral margin; and, after adding, I find a counter at each of of the following numbers, 128; 64; 32; 16; 4; 2, and 1, which form my product, 247.

To Divide. MARK with counters the local dividend in the lateral, and the local divisor in the inferior margin, beginning at the square where a point, descending diagonally from the angle above the highest number of the dividend, would intersect a point ascending perpendicularly from the highest number of the divisor; place a series of counters parallel to the divisor: If this series is equal or inferior in value to the higher number of the dividend subtract it from them; and if otherwise, bring it down one, two, &c. steps and subtract. Repeat the operation till either nothing, or at least a numberless than the divisor, shall remain on the lateral margin. These serieses point to the numbers that form the quotient. For example let it be required to divide 250 by 13. I mark, Fig. IX. the numbers 128; 64; 32; 16; 8; and 2, in the lateral, and 8; 4, and 1 in the inferior margin.

MY first series points to 16. I subtract it from the dividend and find remaining 32; 8, and 2.

MY next series pointing to 4 is too great to be subtracted, for which reason I bring it a step farther down.

AFTER subtracting, there remains 16. In the same manner my third series pointing to 2 I must bring to point to 1; which subtracted, there remains counters on the dividend at 2 and 1. My quotient is therefore 16; 2, and 1, or 19; and 2 and 1, or 3 over.

To extract the square root. MARK the number locally in the lateral margin. From the angle formed by the meeting of the inferior line with the lateral, let a point ascend diagonally till it arrive in a square of the same value with the highest number that can be subtracted from the number whose square root is sought. In this square place a counter, and subtract its value from the number marked in the margin. Form the congruous gnomons, which from the foresaid square have each their upper counter in a line perpendicular and their left hand inferior one in a line parallel to the lateral margin: and subtract their value one by one from the marked remainders. The counters, lying perpendicularly to either of the margins, point out the square root. Let it be proposed, for example, to find the square root of 2209; I mark the numbers 2048; 128, 32, and 1 on the lateral margin. Fig. X. Subtracting the value 1024 of the first counter placed in the table as directed, the remainders are 1024, 128, 32 and 1. From these taking the value 512 and 64 of the first congruous gnomon, there remain 512, 64, 32 and 1. From these taking the value of the second congruous gnomon 256, 64 and 16, there remain 64, 16, 8, 4 and 1: and from these taking the value of the fourth congruous gnomon, 64, 16, 8, 4 and 1, there remains nothing. The square root, as indicated by the direction of the counters in the table, is 32, 8, 4, 2 and 1, or 47.

WHAT is above said will, it is hoped, be sufficient to give a clear idea of the form and use of those of Napier's arithmetical instruments, which seemed to him worthy of being communicated to the public. The reasons on which the different operations are founded, depending upon the construction of the machines and the obvious properties of numbers,

bers, muſt occur to every reader endowed with a moderate ſhare of attention. The hint of the rods, or *virgulæ numeratrices* and of the promptuary which is only an improvement of the rods, ſeems to have been taken from the *Abacus Pythagoricus* or common multiplication table. Napier's acquaintance with cheſs, the moſt ingenious of all games, and at which mathematicians are commonly the beſt players, occaſioned his diſcovery of the *Arithmetica localis.* The Promptuary, at leaſt for multiplication, is greatly preferable to the rods and the cheſs board; for the partial products of two numbers, conſiſting of even ten Figures each, may, by a little practice, be exhibited on that machine in the ſpace of a minute, and no numbers require to be written out, excepting the total product. Had the logarithms remained undiſcovered, the promptuary, in all probability, would have become univerſally familiar to thoſe who were engaged in tedious calculations. But to thoſe who are acquainted with the logarithms, Napier's arithmetical machines and thoſe afterwards invented, a few of which we ſhall enumerate, although the monuments of genius, muſt, in general, be regarded as mathematical curioſities of no uſe.

Perhaps put into the hands of young people learning arithmetic, they might make them fond of that ſtudy.

Shickartus in a letter to Kepler, written in the year 1623, informs him that he had lately conſtructed a machine conſiſting of eleven entire and ſix mutilated little wheels, by which he performed the four arithmetical operations *, Paſcal, in the year 1642, at the age of nineteen,

* Kepl. Epiſt. p. 683.

teen, invented a machine with which all kinds of computations may be made without the pen, without counters, and without the knowledge of any rule of arithmetic. I have not been able to meet with any description of it. It must however have been of a very complicated nature as its author was two years in bringing it to perfection, owing to the difficulty he found to make the workmen understand him thoroughly *. The French writers agree in calling it admirable; † but the name of Pascal perhaps does it more honour than it deserves. This machine is preserved in the cabinet of the king of France and in those of a few others ‡.

THE Marquis of Worcester, a man of genius but a plagiary, mentions in his scantlings of inventions, published in the year 1655, an instrument whereby persons ignorant in arithmetic may perfectly observe numerations and subtractions of all sums and fractions §. Whether he here refers to some of Napier's instruments, to Gunter's scale, of which I shall afterwards speak, or to some invention of his own is uncertain.

ABOUT the year 1670, ** Sir Samuel Moreland contrived two arithmetical instruments; one for addition and subtraction, and the other for multiplicaton, division, and the extraction of the square, cube, and square cube roots, the description of which he published at London, anno 1671 ††.

MUCH

* Les hommes illustres par Perrault-vie de Pascal. † Bayle Chauffepie, Baillet, Perrault, &c.

‡ Pref. Pensees de Pascal. § N° 84. Glag. Edit. 1767.

** Moreland's Instrument of excellent use as well at sea as at land, invented and variously experimented in the year 1670, Lond. 1671. Fol.

†† See also Phil. Transact. N° 94. p. 6048.

MUCH about the ſame time, Mr George Brown, afterwards Miniſter of Kilmaures in Scotland, invented a machine which, in his account of it publiſhed at Edinburgh in the year 1700, he calls the *Rotula Arithmetica*. This machine conſiſts of a box containing a circular plane moveable on a center pin and fixed ring, whoſe circles are deſcribed from the ſame center. The outermoſt circular band of the moveable, and the innermoſt of the fixed, are each divided into a hundred equal parts, and theſe parts are numbered 0, 1, 2, 3, &c. Upon the ring there is a ſmall circle having its circumference divided into ten equal parts, furniſhed with a needle which ſhifts one part at every revolution of the moveable. By this ſimple inſtrument are performed the four arithmetical operations not only of integers but even of decimals finite and infinite *.

SOME time before Mr Brown's little book appeared, Mr Glover had publiſhed a *Roue Arithmetique* ſimilar to the Rotula but not ſo perfect. It would appear however that that gentleman had got ſome hints of the conſtruction of the Rotula from a brother of his own who had been one of Brown's pupils in 1674 †.

IN the year 1725, an inſtrument invented by M. de l'Epine of a more ſimple conſtruction and eaſier in its operations than Paſcal's; in 1730, another by Mr Boiſſendeau, by which calculation is performed without writing; and in 1738 a third by Mr Rauſlin, conſiſting of rods different from thoſe of Napier, were approved of by the French academy ‡.

L SAM

* One of theſe machines is in the library belonging to the faculty of advocates at Edinburgh.

† Rot. Arithm. P[illegible] Paris memoirs for theſe years. Hiſtoire.

Sam Reyer invented, at what time I have not been able to learn, a kind of ſexagenal rods in imitation of Napier's, by which ſexagenary arithmetic is eaſily performed *.

I have an arithmetical machine which came into my poſſeſſion from my uncle George Lewis Erſkine who, though born deaf, by the aſſiſtance of the learned Henry Baker of the Royal Society at London, acquired not only the uſe of ſpeech and the learned languages but a deep acquaintance with uſeful literature. This machine conſiſts of a ſmall ſquare box furniſhed with ſix cylinders moveable round their axes. Upon each of theſe cylinders, which are only Napier's rods, are engraven the ten digits, and their multiples. From a perpetual almanac on the out ſide of the box, it would appear that this machine was conſtructed in the year 1679.

SECTION

* See Chamber's Diction. Article Arithmetic.

SECTION III.

NAPIER'S THEORY OF THE LOGARITHMS *: NEWTON'S IDEAS OF FLUXIONS, BORROWED FROM NAPIER.

I Shall now proceed to unfold the Logarithms, the diſcovery of which has juſtly entitled Napier to the name of the *greateſt Mathematician of his Country*. Let two points, the one in N, and the other in L, (Fig. XI.) having at firſt a ſimilar velocity, move along the indefinite ſtraight lines C N D and K L Δ ; the firſt increaſing its velocity or diminiſhing it according to its diſtance from a fixed point C, and the ſecond preſerving its velocity without augmentation or diminution. Let the former, in a certain time, arrive at any point N′ or n′, and the latter in the ſame time at the point L′ or l′: the ſpace L L′ or L l′ deſcribed by the ſecond moveable point is ſaid to be the Logarithm of the diſtance C N′ or C n′ of the firſt from the fixed point C.

1. THE Logarithm of C N or unity is zero: for the firſt moveable point not having left N, the ſecond has had no time to deſcribe any ſpace.

2.

* The term Logarithm was firſt uſed by Napier after the publication of the Canon in which he uſes the term of *numerus artificialis*.

2. The Logarithms of the terms of a geometrical ſeries are in arithmetical progreſſion: for let N N′, N′N″, N″N‴, &c. or N n′, n′n″, n″n‴, &c. be continual proportionals, they will be deſcribed by the firſt moveable in equal times, and the equal ſpaces L L′, L′L″ L″L‴, &c. or L l′, l′l″ l″l‴, &c. will be deſcribed by the ſecond moveable in the ſame times. Now it is eaſily demonſtrated that C N, C N′, C N″, &c. or C n, C n′ C n″, &c. are in geometrical progreſſion, and it is evident that their reſpective logarithms o, L L′, L L″, &c. or o, L L′, 2 L L′ &c. and o, L l′, L l″, &c. or o, L l′, 2 L l′, &c. are in arithmetical progreſſion.

3. The logarithms of quantities leſs than C N are negative, if thoſe of quantities greater than C N are poſitive; and converſely: for if C n″ C n′, C N C N′, C N″ are continual proportionals, in order that their logarithms 2 L l′, L l′, o, L L′, 2 L L′, &c. may be in arithmetical progreſſion it is neceſſary that the terms on different ſides of zero ſhould have oppoſite ſigns. Hence,

4. The logarithm of any quantity is the ſame with that of its reciprocal, the ſign excepted.

5. The number of ſyſtems of logarithms is infinite: for the ratio of C N to C N′ and L L′ are indeterminate.

6. The logarithms of any one ſyſtem, are to the correſpondent ones of any other, as the value of L L′ in the firſt ſyſtem, is to its value in the

the ſecond. From the 2d propoſition the four following, expreſſed in the language of arithmetic, are eaſily deduced.

7. The logarithm of a product is equal to the ſum of the logarithms of its factors. Thus the logarithm of $CN' \times CN''$ is $LL' + LL'' = LL'''$: for $CN \times CN'' = CN'''$.

8. The logarithm of a quotient is equal to the difference of the logarithms of the diviſor and dividend. Thus the logarithm of $\frac{CN'''}{CN'}$ is $LL''' - LL' = LL''$: for $\frac{CN'''}{CN'} = CN''$.

9. The logarithm of the power of a quantity is equal to the product of the logarithm of that quantity by the index of its power. Thus the logarithm of $\overline{CN'}^3$ is $3\,LL' = LL'''$: for $\overline{CN'}^3 = CN'''$.

10. The logarithm of the root of a quantity is equal to the quotient of the logarithm of that quantity by the index of its root. Thus the logarithm of $\sqrt{CN'''}$ is $\frac{1}{3}LL'$: for $\sqrt{CN'''} = CN'$.

From the 7th and 8th propoſitions the two following are evident.

11. The logarithm of an extreme or mean term of a geometrical proportion, is equal to the difference of the ſum of the logarithms of the means or extremes and the logarithm of the other extreme or mean.

12. If the logarithms of all the primary numbers are known, those of all the composite numbers may be found by simple addition; and if all the latter are known, all the former may be known by simple subtraction.

From the 2nd or the 9th and 10th propositions.

13. The logarithms may be thus defined, *Numerorum proportionalium æquidifferentes comites*; or more properly (as their name, λογων ἀριθμός, imports) *Numeri rationem exponentes*; because they denote the rank, order, or distance, with regard to unity, of every number in a series of continued proportionals of an indefinite number of terms.

14. The logarithm Ll′ of any quantity Cn′ is greater than the difference Nn′ between CN or unity and that quantity, and less than that difference, increased in the proportion of CN to the said quantity: for the velocity of the second moveable describing Ll′ being greater than that of the first describing Nn′ during the same time, Ll′ is greater than Nn′ or CN — C′; and the velocity with which NN′ is described, being greater than that with which Ll′ is described, in an equal time, Ll′ is less than NN′ or CN′ — CN or [since Cn′: CN : : CN′], $[CN - Cn'] \times \frac{CN}{Cn'}$ Hence,

15. If a quantity Cn′ differs infinitely little from CN or unity, its logarithm Ll′ will be equal to $\frac{[CN + Cn'] \times [CN - Cn']}{2\,Cn'}$ the arithmetical,

metical, or to [CN — Cn] $\times \sqrt{\frac{CN}{Cn'}}$ the geometrical mean between its limits above ſtated.

16. The difference l′l″ of the logarithms Ll′ and Ll″ of any two quantities Cn′ and Cn″ is leſs than the difference n′ n″ of theſe quantities increaſed in the proportion of the leſſer Cn″ to CN or unity; and greater than the ſaid difference increaſed in the proportion of the greater Cn′ to CN or unity: for reaſoning in the ſame manner as in the 14th propoſition l′l″ will be found to be leſs than NN′ or [ſince Cn″: CN : : n′ n″:NN′] CN × n′ n″. Hence,

17. If the difference of two numbers Cn′ and Cn″ is infinitely ſmall, the difference of their logarithms will be expreſſed by the arithmetical $\frac{(Cn' + Cn'') \times (Cn' - Cn'') \times CN}{2\, Cn' \times Cn''}$ or the geometrical mean $\frac{Cn' - Cn''}{\sqrt{Cn' \times Cn''}}$ × CN between its limits above ſtated. Beautiful, ingenious and profound! Such is the manner in which Napier conceived the generation of numbers and their logarithms, and ſuch are ſome of their relative properties which naturally flow from it. Thoſe who are acquainted with Newton's manner of explaining the doctrine of fluxions, muſt be ſtruck at its reſemblance to this of our Scotiſh Geometer. This reſemblance, or rather identity, is conſpicuous not only in their ideas but in their very words. The explanation of the firſt definition in the *Canonis mirifici deſcriptio* is in the following terms: *Sit punctus* A, *a' quo ducenda ſit linea* fluxu *alterius puncti*, *qui ſit* B. Fluat *ergo primo* momento B *ab* A in

in C. *Secundo momento à* C *in* D. *Tertio momento à* D *in* E *atque ita deinceps in infinitum defcribendo lineam* A C D E F, &c. *Intervallis* AC, CD, DE, EF *et ceteris deinceps æqualibus, et momentis æqualibus defcriptis*, &c. I the appendix to the *Canonis mirifici conftructio*, under the article *Habitudines Logarithmorum*, he thus expreffes the relation between two natural numbers and the velocities of the increments or decrements of their logarithms; *Ut finus major ad minorem; ita velocitas* incrementi *aut decrementi apud majorem.* What difference is there betwixt this language and that of the great Newton now in ufe $x : y :: \dot{\overline{\text{Log. } y}} : \dot{\overline{\text{Log. } x}}$ * ?

The feeds of the invention of the logarithms were perceived by the ancients as well as by the moderns, upon the revival of fcience in Europe, before the time of Napier. In the elements of Euclid, and in the Arenarius of Archimedes †, thefe great men feem to have been very well acquainted

* See Hutton's Conftruction of Logarithms, p. 42 and 48.

† [In the Arenarius of Archimedes] Without entering in this place, on the repulfion of the received opinion, that this great Mathematician had made the firft ftep towards the knowledge of the Logarithms, I fhall content myfelf with giving the refult of the enquiry, by one of the ableft Mathematicians in the country, to whom I addreffed myfelf, when I firft fet myfelf to produce this work, and who having fuccefsfully illuftrated the difcoveries of the Prince of Englifh Mathematicians, gladly came forward to contribute his fhare to the triumph of our Scotifh Newton.

Archimedes demonftrates a Theorem concerning numbers, made by the mutual multiplication of the terms of a geometrical progreffion; by means of which Theorem the principles of Logarithmic computation may eafily be demonftrated. Archimedes, therefore, had he been furnifhed with tables of Logarithms, would have known how to ufe them: But it appears not, that he was poffeffed of any principles, which could lead him to the formation of Logarithms. He could avail himfelf, indeed, of the indices of the powers of numbers, to abridge the labour of multiplication, as we now avail ourfelves of Logarithms for the like purpofe: But the gulph between this method by the *Natural Indices* acquainted with the correſpondence of an arithmetical to a geometrical progreſſion *.

MICHAEL Stifellius, a German Arithmetician, who flouriſhed about the middle of the ſixteenth century, in his *Arithmetica integra* ſtated the comparifon between the ſeries $\left\{\begin{matrix} 1, & 2, & 4, & 8, & 16, & 32, \\ 0, & 1, & 2, & 3, & 4, & 5, \end{matrix}\right\}$ &c. obſerving that the product or quotient of any two terms of the former correſponded to the ſum, or difference, of the equidiſtant terms of the latter.

dices, and the method of Logarithms, is wider than it may at firſt ſeem. Any Number, not itſelf ariſing from a root, is the root of a diſtinct progreſſion of Powers. Hence there are as many diſtinct progreſſions as there are numbers not actually powers: And in all theſe progreſſions the homologous powers have the ſame exponents or indices. Thus 3 is the exponent of the number 8 in the ſeries of the powers of 2. But 3 is equally the exponent of 27 in the ſeries of the powers of 3; of 125 in the ſeries of the powers of 5; of 343 in the ſeries of the powers of 7: and univerſally of all cubic numbers; ſo 4 is the exponent of all biquadratic numbers; 5 of all quadrato-cubic; and ſo on. A number therefore is not ſufficiently characteriſed by its exponent unleſs it be known to what ſeries of powers it belongs, that is from what root it ariſes. Add to this that many numbers fall into no natural ſeries of powers. This method therefore of computing by the natural indices of powers ariſing from the natural numbers as roots, will only ſerve the purpoſe of rude calculations leading to ſome very general concluſions, and muſt fail in all inſtances in which accuracy is required. Archimedes never thought of conſidering all numbers as expreſſions of proportions, capable of being univerſally included in one general ſeries of ratios, which notion is the true baſis of Napier's great invention, as will be more fully explained hereafter. For the invention in effect was this; that he found a method to raiſe a ſeries of proportionals, in which all numbers ſhould be compriſed, in which every number of conſequence had its own particular exponent, and to find the exponent of any given number, or the number of any given exponent in that univerſal ſeries."

In the courſe of this work, it will be ſufficiently proved, that Napier was as much the firſt to conceive as to execute this wonderful project.

* Thoſe who wiſh to recollect how much we are indebted to the ancients, in this as well as in many other departments of ſciences, will read with pleaſure Mr Duten's Inquiry into the origin of the diſcoveries attributed to the moderns.

Whether Napier ever ſaw or heard of this remark of Stifellius is not known, nor indeed is it of any conſequence; for it cannot fail of preſenting itſelf to any perſon of moderate acuteneſs who happens to be engaged in arithmetical queſtions of this nature where the powers of numbers are concerned. It is not therefore this barren though fundamental remark that can entitle him who firſt mentioned it to the name of the inventor of the logarithms. The ſuperior merit of Napier conſiſts in having imagined and aſſigned a logarithm to any number whatever, by ſuppoſing that logarithm to be one of the terms of an infinite arithmetical progreſſion, and that number one of the terms of an infinite geometrical progreſſion whoſe conſecutive terms differ infinitely little from each other.

The invention of the logarithms has been attributed to Chriſtianus Longomontanus, one of Tycho Brahe's diſciples, and an eminent aſtronomer and mathematician in Denmark. The hackneyed ſtory which gave riſe to this, is told by Anthony Wood in his *Athenæ Oxonienſes* *, and is as follows; "One Doctor Craig, a Scotchman, coming out of "Denmark into his own country, called upon John Neper baron of Merchiſton near Edinburgh, and told him among other things of a new "invention in Denmark (by Longomontanus as 'tis ſaid) to ſave the "tedious multiplication and diviſion in aſtronomical calculations. Neper being ſollicitous to know farther of him concerning this matter, "he could give no other account of it than that it was by proportional "numbers. Which hint Neper taking, he deſired him at his return to "call again upon him. Craig after ſome weeks had paſſed did ſo, and "Neper

* Vol. I. p. 469.

"Neper then ſhewed him a rude draught of what he called *Canon Mi-* "*rabilis Logarithmorum:* which draught with ſome alterations he prin- "ting in 1614, it came forthwith into the hands of our author Briggs "and into thoſe of Will. Oughtred, from whom the relation of this "matter came."

THIS ſtory is either entirely a fiction, or much miſrepreſented. There is no mention of it in Oughtred's writings *. There are no traces of the logarithms in the works of Longomontanus †, who was a vain man and ſurvived Napier twenty nine years ‡, without ever claiming any right to the invention of thoſe numbers, which had for many years been univerſally uſed over Europe.

THE following hypotheſis may perhaps obviate the ſtory of Anthony Wood. Might not Craig, whom reaſon and Tycho Brahe could not diveſt of the prejudices of the Ariſtotelian philoſophy which he had imbibed, on returning to Edinburgh from Denmark, viſit Napier and tell him among other literary news that Longomontanus had invented a method of avoiding the tedious operations of multiplication and diviſion in the ſolution of triangles? After getting the beſt anſwers this doctor could give to Napier's queries relative to this method, I perceive, ſays the baron of Merchiſton, that Longomontanus hath invented, improved, or ſtolen from the *Fundamentum Aſtronomicum*, the Proſthaphæreſis of Raymar: but if you will take the trouble of calling upon me

* Oughtred's Clavis Math. Oxon 1677, &c. † Smith, Briggii vita, and Ward's lives. Art. Briggs.

‡ Voſſius (de Nat. Artium) cited by Ward, places the death of Longomontanus in the year 1647.

me ſome time hence, I will ſhew you a method of ſolving triangles by proportional numbers quite diſtinct from this we have been talking of; which method came into my head ſome ſhort time ago, and will require many years intenſe thinking and labour to bring it to perfection. Accordingly a few weeks afterwards, when Craig returned to Merchiſton, Napier ſhewed him the firſt rude draught of the *Canon Mirificus.* Craig, having occaſion to write very ſoon after to Tycho Brahe, mentioned to him this work without ſaying any thing about its author *.

Justus Byrgius alſo, inſtrument maker and aſtronomer to the Landgrave of Heſſe, a man of real and extraordinary merit, is ſaid by Kepler, in his *Tabulæ Rudolphæ*, to have made a diſcovery of the Logarithms, previous to the publication of the *Canon Mirificus.* The paſſage referred to is as follows: " Apices logiſtici, Juſto Byrgio, multis annis ante editionem Nepeiranam, viam præiverunt ad hos ipſiſſimos logarithmos (i. e. Briggianos) etſi homo cunctator et ſecretorum ſuorum cuſtos, fætum in partu deſtituit, non ad uſos publicos educavit. That is *the accents* (′, ″, ‴, ⁗, &c. denoting minutes, ſeconds, thirds, fourths, &c. of a circular arch) *led Byrgius to the very ſame logarithms* (now in uſe) † *many years before Napier's work appeared: but Juſtus being indolent and reſerved* (or jealous) *with regard to his own inventions, forſook this his offspring* (at or) *in its birth, and trained it not up for public ſervice.*

It

* Nihil autem (writes Kepler to Crugerus) ſupra Nepeiranam rationem eſſe puto: etſi quidem, Scotus quidam, literis ad Tychonem anno 1594, ſcriptis, jam ſpem fecit Canonis illius Mirifici. Kepl. Epiſt. a Gottheb. Hantſch. folio p. 460.

† Thus Byrgius might conceive $Log.\ a^{\circ} = 0$
$Log.\ a' = 1$
$Log.\ a'' = 2$
$Log.\ a''' = 3$ &c a being any number leſs than 60.

It may be obferved that this affair refts on the fingle teftimony of Kepler; but it would perhaps be confidered as a fpecies of herefy to doubt the teftimony of fo great a man. It has been infinuated, however, that from partiality to a countryman he might imagine he faw more than was really to be found in the papers of Byrgius *. Indeed the expreffion, *fœtum in partu deftituit*, gives a colour of truth to the infinuation, and tempts one to think, that Juftus' acquaintance with the logarithms, was much on a par with that of Stifellius. Moreover, it is highly probable, that even the profound and penetrating Kepler might have perufed the manufcripts of Byrgius, without paying any particular attention to his principles of the logarithms, had he himfelf not been previoufly acquainted with Napier's theory of thofe numbers. Neither does it feem probable that Byrgius, had he known its value, could have been fo indolent, fo unreafonably referved, and fo dead to the fenfe of reputation, as to conceal from all the world an invention fo ufeful and fo glorious. We know alfo, that he communicated to his fcholars and others a moft ingenious and eafy method of conftructing the tables of the natural fines †. But fetting all this entirely afide, and granting a great deal more in favour of Byrgius than Kepler's words impute to him; nothing can thereby be detracted from the merit of Napier, who never faw or heard of Byrgius' pretended difcovery of the logarithms; for, by Kepler's own confeffion, *homo cunctator et fecretorum fuorum cuftos*, hoc inventum *non ad ufus publicos educavit.*

O It

* Montucla Hiftoire des Mathematiques.

† This method is unfolded, and dedicated to Juftus Byrgius its inventor by Raymar in his [illegible] [illegible]. See alfo a part of a letter of Rothmannus to Tycho Brahe in [illegible] Tycho [illegible]

It is therefore upon clear and indubitable evidence that, *cum de aliis fere omnibus præclaris inventis plures contendant gentes, omnes Neperum logarithmorum authorem agnoſcunt qui tanti inventi gloria ſolus ſine æmulo fruitur* *; while ſeveral nations contend for almoſt every other famous invention, all agree in recogniſing Napier as the unrivalled author of the logarithms, and as ſolely entitled to the glory of ſo great a diſcovery.

SECTION

* Keil de Log. Præf.

SECTION IV.

NAPIER'S METHOD OF CONSTRUCTING THE LOGARITHMIC CANON.

HAD Napier's principal idea been to extend his logarithms to all arithmetical operations whatever, he would have adapted them to the feries of natural numbers, 1, 2, 3, 4, &c. In that cafe, having confidered the velocity of the two points as continuing the fame for a very fmall fpace of time, after fetting out from N and L (Fig. XI.), he would have taken Nn itfelf as the logarithm of CN + Nn, or Cn. Now as Cn furpaffes CN or unity by a very fmall quantity, it is evident that, when raifed to its fucceffive powers, there will be found in the feveral products fuch as are very near in value to the natural numbers 1, 2, 3, 4, &c. agreeably therefore to the above theory (Sect. III. prop. 9.) Nn being equal to d, and x being a pofitive integer, any natural number may be reprefented by $(1+d)^x$ and its logarithm in Napier's fyftem by x d.

By this formula might the logarithms of all the primary numbers 3, 5, 7, &c. be calculated; from which thofe of all the compofite numbers 4, 6, 8, 9, 10, &c. are eafily deduced by fimple additions (Sect. III. prop. 7.) or by multiplications by 2, 3, 4, 5, &c. (Sect. III. prop. 9).

NAPIER'S

Napier's views were entirely confined to the facilitating of trigonometrical calculations. This is the reaſon of his calculating only the logarithms of the ſines; the log. of any given number being eaſily deduced from theſe by means of a proportion.

In order to effect his purpoſe, he conſidered that the radius, or ſine total, being ſuppoſed to conſiſt of an infinite number of infinitely ſmall and equal parts, all the other ſines would be found in the terms of a geometrical ſeries deſcending from it to infinity; and that the logarithm of the radius being ſuppoſed equal to zero, the logarithms of all the ſeries, beginning with the radius, would be found in the terms of an arithmetical ſeries, aſcending from zero to infinity by ſteps equal to the logarithm of the ratio in which the geometrical ſeries deſcends.

Agreeably to this idea, he ſuppoſes the radius = CN = 10000000, and firſt conſtructs three tables, of which the firſt contains a geometrical ſeries deſcending from the radius to the hundredth term in the ratio of 10000000 to 9999999. It is formed by a continual ſubtracting, from the radius and every remainder, its 10000000th part. The decimals in every term are puſhed to the ſeventh place: a ſpecimen of this table follows.

10000000

10000000 . 0000000
1 . 0000000

9999999 . 0000000
9999999

9999998 . 0000001
9999998

9999997 . 0000003
9999997

9999996 . 0000006
and so on to
9999900 . 0004950

THE second table contains a geometrical series descending from the radius to the fiftieth term, in the ratio of 100000 to 99999, nearly equal to that of the first term 10000000 . 0000000 to the last 9999900 . 0004950 of the first table. It is formed by a continual subtracting, from the radius and every remainder, its 100000th part. The decimals are pushed to the sixth place. A specimen of this table follows.

10000000 . 0000000
100 . 000000

9999900 . 000000
99 . 999000

9999800 . 001000
99 . 998000

9999700 . 003000
99 . 997000

9999600 . 006000
and so on to
9995001 . 222927

RADICAL TABLE.

FIRST COLUMN.		SECOND COLUMN.	
NATURAL.	ARTIFICIAL.	NATURAL.	ARTIFICIAL.
10000000 . 0000	0	9900000 . 0000	100503 . 3
9995000 . 0000	5001 . 2	9895050 . 0000	105504 . 6
9990002 . 5000	10002 . 5	9890102 . 4750	110505 . 8
9985007 . 4987	15003 . 7	9885157 . 4237	115507 . 1
9980014 . 9950	20005 . 0	9880219 . 8451	120508 . 3
and so on to	and so on to	and so on to	and so on to
9900473 . 5780	100025 . 0	9801468 . 8423	200528 . 2

and so on to

COLUMN 69.

NATURAL.	ARTIFICIAL.
5048858 . 8900	6834225 . 8
5046333 . 46 5	6839227 . 1
5043811 . 2932	6844228 . 3
5 41289 . 3879	6849229 . 6
3038763 . 7435	6854230 . 8
and so on to	and so on to
4998609 . 4034	6934250 . 8

The numbers and logarithms in the above table, coinciding nearly with the natural and logarithmic sines of all the arcs from 90° to 30°, he was enabled, by means of prop. 16. or 17. and a table of the natural sines, to calculate the logarithmic sines to every minute of the last 60° of the quadrant.

In order to obtain the logarithms of the sines of arcs below 30°, he proposes two methods.

The first is this. He multiplies any given sine of an arc less than 30° by the number 2, 10, finding the logarithms of the numbers 2 and

and 10 by means of the radical table, or takes ſome one of the compounds of theſe ſo as to bring the product within the compaſs of the radical table. Then having found, in the manner before deſcribed, the logarithm of this product, he adds to it the logarithm of the multiple he had made uſe of; the ſum is the logarithm ſought.

The ſecond method is derived from a property of the ſines which he demonſtrates. The property is this: Half the radius is to the ſine of half an arc, as the ſine of the complement of half that arc is to the ſine of the whole arc. Hence, as is evident from a foregoing prop. that the logarithm of the ſine of half an arc may be had by ſubtracting the logarithm of the ſine of the complement of half that arc from the ſum of the logarithms of half the radius and of the ſine of the whole arc.

By this ſecond method, which is much eaſier than the firſt, the logarithms of the ſines of the arcs below 45° may be obtained; thoſe above 45° having been calculated by help of the radical table.

The logarithms of the ſines to every minute of the quadrant being found by the ingenious methods above deſcribed, the logarithms of the tangents were eaſily deduced by one ſimple ſubtraction of the logarithm of the ſine of the complement from that of the ſine for each arc. The logarithm of the radius, which ſo frequently occurs in trigonometrical ſolutions, having been very advantageouſly made equal to zero, the logarithms of all the tangents of arcs below 45° and of all the ſines muſt have a different ſign from that of the logarithms of all the tangents of arcs above 45°. Napier chose the positive sign for the former which he calls *abundantes*, and the negative for the latter which he calls *defectivi*.

The arrangement of the numbers in Napier's logarithmic table, is nearly the same with that neat one which is still in use. The natural and logarithmic sines and the logarithmic tangent of an arc and of its complement stand on the same line of the page. The degrees are continued forwards from 0° to 44° on the top, and backwards from 45° to 90° on the bottom of the pages. Each page contains seven columns; the minutes descend from 0′ (to 30′ or from 30′) to 60′ in the first, and from 60′ (to 30′ or from 30′) to 0′ in the last of these columns. The natural sines of the arcs, on the left and on the right hand, occupy the second and sixth column, and their logarithms the third and fifth respectively. The fourth column contains the logarithms of the tangents which are taken positively if they refer to the arcs on the left, and negatively if they refer to the arcs on the right hand. A specimen of this table is here annexed.

Gr.

Gr. 44 mi.	SINUS.	LOGARITHMI.	DIFFERENTIA.	LOGARITHMI.	SINUS.	
30	7009093	3553767	174541	3379226	7132504	30
31	7011167	3550808	168723	3382085	7130465	29
32	7013241	3547851	162905	3384946	7128225	28
33	7615314	3544895	157087	3387808	7126385	27
34	7017387	3540941	151269	3390672	7124344	26
35	7019459	3538989	145451	3393537	7122303	25
36	7021530	3536038	139633	3396406	7120261	24
37	7023601	3533089	133814	3399275	7118218	23
38	7025671	3530142	127996	3402146	7116175	22
39	7027741	3527197	122178	3405019	7114131	21
40	7029810	3524243	116359	3407894	7112086	20
41	7031879	3521311	110541	3410770	7110041	19
42	7033947	3518371	104723	3413648	7107995	18
43	7036014	3515432	98904	3415528	7105949	17
44	7038081	3512495	92886	3419409	7103902	16
45	7040147	3509560	87268	3422292	7101854	15
46	7042213	3506626	81450	3425176	7099806	14
47	7044278	3503694	75632	3428062	7097757	13
48	7046342	3500764	69824	3430940	7095708	12
49	7048406	3497835	64006	3433829	7093658	11
50	7050469	3494905	58178	3436730	7091607	10
51	7052532	3491983	52360	3439623	7089556	9
52	7054594	3489060	46543	3442517	7087504	8
53	7056655	3486139	40726	3445413	7085452	7
54	7058716	3483219	34908	3448311	7083399	6
55	7060776	3480301	29090	3451211	7081345	5
56	7062386	3477385	23273	3454112	7079291	4
57	7064695	3474470	17455	3457015	7077236	3
58	7066953	3475557	11637	3459920	7075181	2
59	7069011	3468645	5818	3462827	7071125	1
60	7071068	3465735	0	3465735	7071068	0
						45 mi. Gr.

In the Appendix to the *Canonis mirifici conſtructio*, Napier delivers three other methods of computing the logarithms; but as theſe methods are generally better adapted to the conſtruction of a ſpecies of logarithms different from that I have deſcribed, I ſhall poſtpone the account of them to the next ſection.

The

The ingenious method by which Napier conſtructed the radical table is almoſt peculiar to the ſpecies of logarithms it contains: It does not ſeem, however, to be ſuſceptible of all the accuracy one would wiſh: for, notwithſtanding the many precautions he had taken, particularly in puſhing his numbers to ſeveral decimal places, the logarithms in his canon often differ from the truth by ſeveral units in the laſt figure. Of this he himſelf was appriſed by finding different reſults from the two methods of determining the logarithmic ſines of arcs under 30°. In order to remedy this defect, he propoſes adding another zero to the radius; by which means, in purſuing this ſame method, the logarithms of the ſines might be obtained true to an unit in the eight figure.

SECTION

SECTION V.

THE COMMON LOGARITHMS DEVISED BY NAPIER AND PREPARED BY BRIGGS, AND THE METHODS PROPOSED BY NAPIER FOR COMPUTING THEM.

ONE capital difadvantage attending the fpecies of logarithms which firft occurred to Napier, arifes from the difference between the fign of the logarithms of the tangents of arcs greater than 45° and the fign of the logarithms of the fines of all the arcs of the quadrant.

THIS defect was eafily remedied by fuppofing the fmalleft poffible fine equal to 1 and its logarithm 0; as in this cafe, the logarithms of all the fines and tangents of every arc in the quadrant would have the fame fign. But, if the fame fpecies of logarithms is made ufe of, the logarithm of the radius, which occurs fo frequently in trigonometrical folutions, would be a number difficult to be remembered. More, therefore, would be loft than gained by this alteration. What fpecies of logarithms will free us from a difference in the figns, and at the fame time afford a logarithm of the radius that fhall be eafily remembered and eafily managed? It was this very queftion, in all probability, that led to the common logarithms, which, of all others, are the beft adapted to our modern arithmetical notation. This ſyſtem of logarithms has for its baſis 1 as logarithm of the ratio of 10 to 1 : ſo that the powers 1, 10, 100, 1000, &c. of the number 10 have their reſpective logarithms 0, 1, 2, 3, &c. * Here, by the bye, it may be obſerved, that not only Napier's manner of conceiving the generation of the logarithms, but his having computed that ſpecies of logarithms, which has been diſcribed, before the common logarithms occurred to him, *is a convincing proof of his not taking the hint of the logarithms from the remark of Stifellius*, formerly mentioned. I think it is even beyond doubt that Napier, in common with all other arithmeticians acquainted with the Arabic, or rather Indian figures, had obſerved that the product of any power of the number 10 by any other power of that number, was formed by joining or adding the zeros in the one to thoſe in the other; and that the quotient of any one power of that number by any other, was formed by taking away or defacing a number of zeros in the dividend equal to the number of zeros in the diviſion; and all this without thinking that he was, at that time, making the fundamental remark of the logarithms. Nor will this ſeem at all ſurpriſing to thoſe who are acquainted with the hiſtory of ſcience and of the human mind. It is ſeldom that we directly arrive at truth by the moſt natural and eaſy path.

Perhaps

* We have ſeen Sect. III. that in Napier's ſyſtem the velocity of the two moveable points in N and L Fig. XI. is equal and that the logarithm (Ll)× of any number (CN+Nn)× or 1.0000000,1)× is nearly equal to (Nn)× or [.0000000,1]× In the common ſyſtem the velocity at L is leſs than the half of the velocity at N; and the logarithm Ll of the number [CN+Nn]× [or 1.0000000,1]× is nearly equal [0.4342945] Nn × x or [0.0000000,0434,2945]×: for in making this ſuppoſition the logarithm of 10, is found to be 1. The logarithms therefore in Napier's ſyſtem are to the correſpondent ones in the common ſyſtem as 1 is to 0. 4342945 or, what is the ſame thing the common logarithms are to thoſe of Napier as 1 is to 2.3025851.

Perhaps the ſtrongeſt mark of the greatneſs of Napier's genius is not his inventing the logarithms, but his manner of inventing them. But to return; In this new ſyſtem the radius was made equal to the 10th power 10000000000 of the number 10, of which the logarithm in the new ſcale is 10. The diviſion of the radius into ſo great a number of parts, render the ſine of the ſmalleſt ſenſible arc greater than 1, of which the logarithm is zero: conſequently, the logarithms of all the ſines and tangents of the arcs of the quadrant, being on the ſame ſide of zero, have the ſame ſign.

WITH regard to the logarithm of the radius, its being eaſily managed is ſufficiently obvious.

THUS in our common logarithms the diſadvantages of Napier's ſyſtem are avoided, whilſt its advantages are retained and united to ſeveral others. Of theſe additional advantages in the common canon, the moſt capital is, that the units in the firſt figure (to which Briggs gave the name of characteriſtic) of the logarithm are fewer by one than the figures of the number to which that logarithm correſponds.

WHETHER Napier, or Briggs, *firſt* imagined this new ſpecies of logarithms, is a queſtion which the learned do not ſeem as yet to have perfectly decided.

THE only evidence we have on which a deciſion can be grounded, is contained in the following particulars.

I,

I. In a letter to Usher afterwards Archbishop of Armagh dated the 10th of March 1615, the year after the publication of Napier's Canon. Briggs writes thus *, "Napier lord of Merchiston hath set my head "and hand at work with his new and admirable logarithms: I hope "to see him this summer if it please God; for I never saw a book "which pleased me better, and made me more wonder."

II. In the dedication of his Rabdologia, published 1617, Napier has the following words, "Atque hoc mihi fini proposito, logarithmorum "canonem a me longo tempore elaboratum superioribus annis edendum "curavi, qui rejectis naturalibus numeris, et operationibus quæ per "eos fiunt, difficilioribus, alios substituit idem præstantes per faciles "addtiones, substractiones, bipartitiones, et tripartitiones. Quorum "quidem logarithmorum *speciem aliam multo præstantiorem nunc etiam in-* "*venimus*, et creandi methodum, una cum eorum usu (si Deus lon- "giorem vitæ et valetudinis usuram concesserit) evulgare statuimus: "ipsam autem novi canonis supputationem, ob infirmam corporis nostri "valetudinem, viris in hoc studii genere versatis relinquimus: impri- "mis vero doctissimo viro D. Henrico Briggio Londini publico Geo- "metriæ Professori, et amico mihi longe charissimo".

III. In the preface to the *logarithmorum chilias prima*, a table of the common logarithms of the first thousand natural members, Briggs expresses himself in the following terms; "Why these logarithms differ "from those set forth by their illustrious inventor, of ever respectful "memory, in his *canon mirificus*, it is to be hoped, his posthumous work "will shortly make appear."

IV.

* The life of Archbishop Usher and his correspondence, by Richard Par, D. D. 1686. folio, page. 36.

IV. In the preface the Arithmetica Logarithmeca *, there is the following paragraph, " Quod hi logarithmi diverſi ſunt (writes Briggs) ab " iis quos clariſſimus vir baro Merchiſtonii in ſuo edidit canone mirifi- " co, non eſt quod mireri, enim meis auditoribus Londini publice in " Collegio Greſhamenſi horum doctrinam explicarem; animadverti mul- " to futurum commodius, ſi logarithmus ſinus totius ſervætur o zero " (ut in canone mirifico) logarithmis autem partii decimæ ejuſdem ſinus " totius, nempe ſinus 5 grad. 44 min. 21. ſecund. eſſet 1.00000,00000: " atque ea de re ſcripſi ſtatim ad ipſum. Authorem, et quamprimum " hic anni tempus, et vacationem a publico docendi munere licuit, " profectus ſum Edinburgum; ubi humaniſſime ab eo acceptus hæſi " per integrum menſem. Cum autem inter nos de horum mutatione " ſermo haberetur; *Ille ſe idem dudum ſenſiſſe*, et capuiſſe dicebat: ve- " runtamen iſtos, quos jam paraverat, edendos curaſſe, donec alios, ſi " per negotia et valetudinem liceret, magis commodos confeciſſet. Iſ- " tam autem mutationem ita faciendam cenſebat, ut o eſſet logarithmus " unitatis et 1,00000 . 00000 ſinus totius: quod ego longe commodiſſi- " mum eſſe non potui non agnoſcere". " Capi igitur ejus hortatu, re- " jectis illis quos antea paraveram, de horum calculo ſerio cogitare, et " ſequenti æſtate iterum profectus Edinburgum, horum quos hic exhi- " beo præcipuos, illi oſtendi. Idem etiam tertia æſtate libentiſſime fac- " turus, ſi Deus illum nobis tamdiu ſuperſtitem eſſe voluiſſet †."

It may here be obſerved, that the manner in which Briggs propoſed the application of the common logarithms to trigonometrical purpoſes,

S did

* Publiſhed in 1624.

† Ulacg in his title page to his edition of Brigg's log. writes to the ſame purport. " Hos numeros " primis invenit clariſſimus vir Joannes Neperus Baro Merchiſtonii; *eos autem ex ejuſdem ſententia, mu-* " *tavit*, eorumque ortum et uſum illuſtrait Henricus Briggius".

did not at all tend to obviate the chief disadvantage of Napier's Canon: For according to Briggs' idea the sign of the logarithms of the sines and the tangents less than the radius must be the opposite of the sign of the logarithms of the tangents greater than the radius. It seems probable, therefore, that Briggs had been led to the common logarithms in endeavouring to get rid of the indirect method of finding the logarithms of the natural numbers by means of Napier's logarithmic Canon.

FROM the extracts above given it appears that the common logarithms had occurred to Napier before they occurred to Briggs: For the modesty and integrity of Napier's character put beyond dispute the truth of what he mentioned to Brigg's at their first meeting, and to the Earl of Dunfermline in the dedication of the Rabdologia. But if the having first communicated an invention to the world be sufficient to entitle a man to the honour of having first invented it, Briggs has a better title than Napier to be called the inventor of this happy improvement of the logarithms *. For Briggs mentioned it to his pupils in Gresham College before the publication, in 1616, of Edward Wright's translation of the *Canon Mirificus*, in the Preface to which Napier gave the first notice of this improvement. With regard to the passage in the preface to the *Chilias prima* published after Napier's death, where Briggs seems to require an acknowledgment from the editor of the Canonis mirifici constructio, that *he* had also imagined the new logarithms; the oversight or fault lies at the door of Napier's son and not at his own. Had Napier lived to publish his last mentioned work, it is hardly possible to entertain a shadow of doubt, but that he would have done ample

* Hutton Math. Tab.

ple juſtice to Briggs in this particular. Napier and Briggs had a reciprocal eſteem and affection for each other, and there is not the ſmalleſt evidence of there having exiſted, in the breaſt of either, the leaſt particle of jealouſy; a paſſion unbecoming and diſgraceful in a man of merit.

We ſhall diſmiſs this affair with obſerving, 1. That after the invention of the logarithms, the diſcovery of the beſt ſpecies of logarithms was no difficult affair: 2. That the diſcovery of the common logarithms at that time, was a fortunate circumſtance for the world, as there are few poſſeſſed of ingenuity and patience ſufficient for the conſtruction of ſuch extenſive and accurate tables as are thoſe of Briggs' *Arithmetica logarithmica*; and 3. That the invention of the new ſpecies of logarithms is far from being equal to ſome other of Briggs' inventions.

We come now to give a very brief deſcription of thoſe other methods of conſtructing the logarithms, propoſed by Napier in the appendix to his poſthumous work.

The firſt of theſe methods is the following: The logarithm of 1 being ſuppoſed 0, and the logarithm of 10 1 followed by any number of zero, 10000000000 for example; this laſt logarithm and the ſucceſſive quotients divided (ten times) by the number 5 will give theſe (ten) logarithms 2000000000, 400000000, 80000000, 16000000, 3200000, 640000, 128000, 25600, 5120, 1024 to which the reſpective correſpondent numbers may be found by extracting the 5th root, the 5th root of the 5th root, the 5th root of the 5th root of the 5th root, &c. of the

the number 10. Then the laſt logarithm 1024 and the ſucceſſive quotients divided (ten times) by the number 2, will give theſe (ten) logarithms 512, 256, 128, 64, 32, 16, 8, 4, 2, 1, &c. to which the reſpective correſpondent numbers may be found, by extracting the ſquare root, the ſquare root of the ſquare root, the ſquare root of the ſquare root of the ſquare root, &c. of the number (found as above directed) correſponding to the logarithm 1024. By addition theſe (twenty) logarithms, and by multiplication their reſpective natural numbers ſerve for finding a great many other logarithms and their numbers.

The ſecond method is this: The logarithms (0 and 10000000000 for example,) of any two numbers 1 and 10 being given, the logarithm of any intermediate number (2 for example) may be found by taking continually geometrical means, firſt between one of theſe numbers (10) and this mean, then between the ſame number (10) and the laſt mean, and ſo on till there be found the number (2) wanted; of which the logarithm will be the correſponding arithmetical mean (3010299957) between the two given logarithms (0 and 10000000000).

The third method is as follows: Suppoſe the common logarithm of a number not an integral power of 10 (2 for inſtance) find the number of figures in the 10th, 100th, 1000th, &c. power of that number: The ſucceſſive numbers of figures (4, 31, 302, 3011, &c.) in theſe powers (2^{10}, 2^{100}, 2^{1000}, 2^{10000}, &c.) will always exceed by leſs than unity, but continually approach to the logarithm [30102999566, &c.] required.

The

The firſt of theſe methods is very operoſe, and by itſelf inſufficient for conſtructing a complete logarithmic canon. The other two are much preferable. The laſt is particularly well adapted for finding the logarithms of the lower prime numbers: For, ſince the number of figures in the product of two numbers, is equal to the ſum of the number of figures in each factor; except the product of the firſt figures in each factor be expreſſed by one figure only, which often happens; a few of the firſt, or left hand figures of the conſecutive tenth powers of the given number, will ſuffice for finding the number of figures in theſe powers.

This laſt method depends on the diſtinguiſhing property of the common logarithms, which is, as was formerly obſerved, that the units in [x] the rational logarithm of a number [10^x] are one fewer than the number of figures in that number [10^x]. Whence it follows, that the units in the irrational logarithm of any other number are not quite one fewer than the number of integral figures in this other number. Now, as in a ſeries of continued proportional numbers, the reſult of any two terms is the ſame, if one of the terms is raiſed to the power indicated by the exponent of the other, or if this other is raiſed to the power indicated by the exponent of the firſt; any number raiſed to the power indicated by the logarithm of 10 is equal to 10 raiſed to the fewer indicated by the logarithm of that number. If, therefore, [the logarithm of 10 being 10000, &c.] Y is any number not an integral power of 10 and y its logarithm, we ſhall have $Y^{10000,\&c.} = 10^y$, and the number of figures in $Y^{10000,\&c.}$ will exceed y by leſs than 1.

T SECTION.

SECTION VI.

THE IMPROVEMENTS MADE ON THE LOGARITHMS.

THE improvements that have been made upon the logarithms after the death of their inventor, regard the theory, the methods of conſtruction, the accuracy, extenſiveneſs, and form of the tables of theſe numbers.

HOWEVER ingenious and beautiful Napier's manner of delivering the theory of the logarithms is, it muſt be acknowledged that it labours under one capital impropriety—treating geometrically a ſubject which properly belongs only to arithmetic. Senſible of this, Kepler *, Nicolas Mercator †, Halley ‡, Cotes ‖, and other mathematicians of the firſt note, have treated the theory of the logarithms in a different and truly ſcientific manner. Their ideas are founded on the definition of the logarithms—*Numeri rationem exponentes;* which, although it is not expreſsly Napier's, is eaſily deducible from his theory. Thus, in a geometrical progreſſion, having any finite number c greater than unity for it's baſis, the exponent x is the logarithm of the ratio of the number c^x to c^0 or

* Chilias Logarithmorum 1624. Tab. Rudolph. 1627. † Logarithmo technia, 1668.
‡ Phil. Trans. 1695. ‖ Harmonia Menſurar. 1722.

or unity: And, if the quotient of two quantities is taken as the meaſure of their ratio, the definition is rendered more ſimple, and x will be the logarithm of c^x. Upon this principle is founded the analytical theory of the logarithms in the appendix.

It was chiefly by the two laſt methods, deſcribed in the foregoing ſection, that Briggs conſtructed his logarithms. He invented alſo an original method of conſtructing logarithms by means of the firſt, ſecond, third, &c. differences of given logarithms. How he came by it is not known. He deſcribes it in his arithmetica logarithmica and there is a demonſtration of it in Cotes's Harmonia, in Bertrand's Mathematiques, and in the works of a great many other authors.

Edmund Gunter, Profeſſor of Aſtronomy in Greſham College, who was the firſt that publiſhed a table of the logarithmic ſines and tangents of that kind which Napier and Briggs had laſt agreed on, applied, in the year 1623, or 1624, the logarithms to a ruler which bears his name. This ſcale is of very great uſe in Navigation, and in all the practical parts of geometry where much accuracy is not required. On the account of this logarithmical invention, Gunter, after Napier and Briggs, has the beſt claim to the public gratitude.

After Napier's death almoſt fifty years elapſed before the inventions of the expreſſions of the logarithms by infinite ſerieſes. Of theſe the three following, from which a great number of others are eaſily derived, were the firſt. *

Logarithm

* Appendix an. th. log.

Logarithm of $(1+a)=a-\frac{1}{2}a^2+\frac{1}{3}a^3+\&c.$ - - - - - X

Logarithm of $(1-a)=-a-\frac{1}{2}a^2-\frac{1}{3}a^3-\&c.$ - - - - Y

Logarithm of $\left(\frac{1+a}{1-a}\right)=a+\frac{1}{3}a^3+\frac{1}{5}a^5+\&c.$ - - - - - Z

These formulæ X, Y, and Z will converge the more quickly in proportion as a is ſuppoſed leſs than unity; and the ſum of a few terms will generally ſuffice. They are the values of Napier's logarithms, but will repreſent every ſpecies of logarithms by being multiplied by an indeterminate quantity u, which is called the *modulus* of the ſyſtem.

The formula X was invented by Nicholas Mercator in the year 1667, and publiſhed in his *Logarithmotechnia* the year following. Gregory of St Vincent, about twenty years before, had ſhewn that one of the aſymptotes of the hyperbola being divided in geometrical progreſſion, its ordinates parallel to the other aſymptote are drawn from the point of diviviſion, they will divide into equal portions the ſpaces contained between the aſymptote and the curve: From this it was afterwards pointed out by Merſennus, that, by taking the continual ſums of thoſe parts there would be obtained areas in arithmetical progreſſion correſponding to abſciſſes in geometrical progreſſion, and conſequently that theſe areas were analogous to a ſyſtem of logarithms *. Wallis, after this, had remarked that the ordinate correſponding to the abſcis a, counted on the aſymptote of the equilateral hyperbola from a diſtance equal to the ſemi-axis 1, is equal to $\frac{1}{1+a}$; and he had demonſtrated, in his *Arithmetica infinitorum* publiſhed in 1655, that the ſum of $1^m+2^m+3^m+\&c.$ --- $+a^m$ (a repreſenting a finite quantity divided into an infinite num-

U ber

* Hutton's Math. Tab.

ber of equal parts) is equal to $\frac{a^{m+1}}{m+1}$ *. With theſe data Mercator ſet himſelf to find the area correſponding to the abſciſs a, or, what is the ſame thing, the logarithm of $(1+a)$, which he happily accompliſhed by firſt developing, in the manner now commonly practiſed, the fraction $\frac{1}{1+a}$ into $1-a+a^2-a^3+$ &c. which had not been attempted before: then, ſuppoſing a equal ſucceſſively to 1, 2, 3, 4, &c. - - - - a, and laſtly, taking ſucceſſively the ſums of all the zero, firſt, ſecond, third, &c. powers of theſe numbers †.

In the ſame year 1668 James Gregory, in his *Exercitationes Geometricæ*, gave a demonſtration of Mercator's formula for the quadrature of the hyperbola different from his. He demonſtrated the formula Y and found the formula Z by ſubtracting Y from X. He found too the the value of log. $(1-a^2)=-a^2-\frac{1}{2}a^4-\frac{1}{3}a^6-$ &c. by adding Y to X: but this formula may be looked on as a ſolecifm when applied to numbers: for the ſame reſult will be obtained by ſuppoſing a to be a ſquare, in the formula Y, and even a more general reſult may be obtained by ſuppoſing a to be any power of a number.

Sir Iſaac Newton, by his general method of the quadrature of curves, greatly ſimplified that of the quadrature of the aſymptotic ſpaces of the equilateral hyperbola. The ordinate, (being as before $=$) $\frac{1.}{1+a}$ multiplied by $\dot{a}$ the fluxion of the abſciſs, becomes the fluxion of the correſponding aſymptotic area: This product, developed in the manner of Mercator, is $\dot{a}-a\dot{a}+a^2\dot{a}-a^3\dot{a}+$ &c. Taking the fluent of each term of

* Montucla Hiſt. de Math. † Appendix, Hyperbola.

of this feries gives the fluent of the area that is the logarithm of $1+x$ equal to X as before. It appears, from a letter of Newton's to Oldenburgh, that Newton had difcovered the quadrature of the hyperbola by infinite but perhaps not general feriefes, before the publication of the *Logarithmotechnia* *. Something of the fame kind had alfo been difcovered by Lord Brouncker †.

The areas of the equilateral hyperbola, as above defcribed, exhibiting the logarithms of Napier's fyftem, occafioned the appellation *hyperbolic* to his logarithms. It is difficult to account for the propriety of this epithet to Napier's logarithms; fince not only the afymptotic areas of equilateral but thofe of any other hyperbola may be made to reprefent every poffible fpecies of logarithms, by fuppofing, in the fame hyperbola, the origin of the abfciffes on the one afymptote at different diftances from its interfection with the other. Thus the afymptotic areas of the equilateral hyperbola will reprefent the common logarithms, if the origin of the abfciffes is taken at the point on the afymptote where the ordinate is $u = 0 . 43429$ &c. the diftance of that point from the other afymptote being greater than the femi-axis but equal to 1 ‡. But if the origin of the abfciffes is taken equidiftant from the fummit of the hyperbola and the interfection of the afymptotes, the afymptotes of the hyperbola, whofe areas reprefent the common logarithms, are inclined to each other about 25°. 44′, of which the fine is $u = 0.43429$ &c. ‖

The formulæ X and Y have alfo been deduced from the logarithmic §—a curve whofe abfciffes are the logarithms of its ordinates or converfely

* Wallifii Opera. vol. 3. p. 634 and feq. cited by Hutton. † Montucla. ‡ Appendix, Hyperbola. ‖ Hutton's Math. Tab. § Encyclopedie au mot Logarithmique. Appendix, Logarithmic.

converſely *. This curve is ſaid to have been invented by Edmund Gunter †; but its properties, ſome of which are very remarkable, do not ſeem to have been much known and attended to till the time of Huygens, who enumerates them in his *Cauſa gravitatis.* It was conſidered afterwards by Leibnitz, Bernoulli, l'Hopital, and a great many others. The manner in which it is treated by John Keill in the tract on the logarithms ſubjoined to his edition of Euclid, facilitates very much the conception of theſe numbers. In the Appendix the reader will find this curve treated in a new manner, with an enumeration of ſome new properties.

The ſame formulæ X and Y are eaſily deduced by the fluxionary method from Neper's generation of the logarithms. From what is ſaid in a foregoing ſection it is evident that (Fig. XI.) the velocity of the firſt moveable at the point N is to its velocity at the point N′ as CN is to CN′; but the velocity of the firſt moveable at the point N is the ſame with the velocity of the ſecond moveable point at (any point of K△) L′: therefore, in the language of fluxions, if NN′ $= a$, $\dot{\overline{\text{Log.}\ (1+a)}} : \dot{a} :: 1 : 1+a$, therefore $\dot{\overline{\text{Log.}\ (1+a)}} = \frac{\dot{a}}{1+a} = \dot{a} - a\dot{a} + a^2\dot{a}$—&c. therefore Log. $1 + a = a - \frac{a^2}{2} + \frac{a^3}{3}$—&c. If the points n′ and l′ are taken, it may be ſhewn in the ſame manner that Log. $[1 - a] = -a - \frac{a^2}{2} = \frac{a^3}{3}$—&c.

In the year 1695, Edmund Halley greatly improved the theory of the logarithms, by deriving the ſerieſes for their conſtruction from the principles

* Appendix, Logarithmic. † Montucla.

them, form the logarithm of the number arifing from the junction of the digit at top or bottom to the figures in the firft column, correfponding to faid four figures. When the laft of the three firft figures of a logarithm, correfponding to a number formed by figures in the firft column and a fignificant digit at top, is found augmented by unity, thefe three figures, together with the correfpondent fours, are moved a line downwards; by this means one avoids the miftaking one three figures for another, which, without fpecial care, muft often be the cafe in ufing Sherwin's, Gardiner's or Hutton's Tables. The laft column contains the differences of the confecutive logarithms, together with the proportional parts correfponding to the nine digits. With thefe proportional parts one can compute by the eye alone the logarithms, not in the table, of all the numbers lefs than 1029600, and, with very little trouble more, thofe of all numbers lefs than 10296000, as exactly as eight places of figures can exhibit them. In the table of the logarithmic fines and tangents, the degrees and minutes are difpofed nearly in the fame manner as in Napier's Table. Each page contains eleven columns. In the firft and laft are the minutes. In the fecond and laft but one are the feconds 0, 10, 20, 30, 40, 50, 0, and 0, 50, 40, 30, 20, 10, 0, of which the firft and laft zeros are in the fame line with and the reft between each fucceeding minute. In the third, fifth, feventh and ninth columns are the logarthmic fines or cofines, cofines or fines, tangents or cotangents, and cotangents or tangents, according as they refer to the degrees at top and the minutes and feconds in the firft and fecond column, or to the degreees at the bottom and the minutes and feconds in the laft penult columns. The other three columns contain the differences of thefe logarithms. The above defcription will become perfectly intelligible by infpecting the following fpecimens of thefe Tables.

TAB. DES LOG. DES NOMB. NAT.

N. 14800 L. 170

N	0	1	2	3	4	5	6	7	8	9
1408	170.2617	2911	3204	3497	3791	4084	4377	4670	4964	5257
81	5551	5844	6137	6430	6723	7017	7310	7603	7896	8189
82	8482	8775	9068	9361	9654	9947				
	171.						0240	0533	0826	1119
83	1412	1704	1997	2290	2583	2876	3168	3461	3754	4046
84	4339	4632	4924	5217	5509	5802	6095	6387	6680	6972
1485	7265	7557	7849	8142	8434	8727		9311	9604	9896
							9019			
86	172.0188	0480	0773	1065	1357	1649	1941	2233	2526	2818
87	3110	3402	3694	3986	4278	4570	4862	5154	5446	5737
88	6029	6321	6613	6905	7197	7488	7780	8072	8364	8655
89	8947	9239	9530	9822						
	173.				0113	0405	0697	0988	1280	1571
1490	1863	2154	2446	2737	3028	3320	3611	3903	4194	4485
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
.	.									
N	0	1	2	3	4	5	6	7	8	9

diff. part.

5	147
6	176
7	206
8	235
9	265

293	
1	29
2	59
3	88
4	117
5	147
6	176
7	205
8	234
9	264

292	
1	29
2	58
3	88
4	117
5	146

TAB. DES LOG. DES SIN. ET TANG.
12 DEC.

I	″	sin.	diff.	co-sin.	diff.	tang.	diff. com.	co-tang.	″	I
20	0	9.32959.88		9.98985.97		9.33973.91		0.66026.09	0	48
			962		46		1009			
	10	9.32969.50	963	9.98985.51	47	9.33984.00	1009	0.66016.00	50	
	20	9.32979.13	962	9.98985.04	46	9.33994.09	1008	0.66005.91	40	
	30	9.32988.75		9.98984.58		9.34004.17		0.65995.83	30	
			963		46		1008			
	40	9.32998.38	962	9.98984.12	46	9.34014.25	1008	0.65985.75	20	
	50	9.33008.00	961	9.98983.66	46	9.34024.33	1008	0.65975.67	10	
21	0	9.33017.61		9.98983.20		9.34034.41		0.65965.59	0	39
			962		46		1008			
	10	9.33027.23	961	9.98982.74	46	9.34044.49	1007	0.65955.51	50	
	20	9.33036.84		9.98982.28		9.34054.56		0.65945.44	40	
	.	.	.	.	.	.		.		
	.	.	.	.	.	.		.	.	
.	.	.	.	.	.	.		.	.	.
.	.	.	.	.	.	.		.	.	.
.	.	.	.		.				.	.
I	″	co-sin.	diff.	sin.	diff.	co-tang.	diff.	tang.	″	I

77 DEG.

THESE

THESE Tables, which are executed with a new and elegant type on good paper, form a ſmall octavo volume. There is every probability in favors of their correctneſs. They are copied from the London edition of Gardiner printed in 1742, which is in the higheſt eſtimation for that quality. Meſſieurs Callet. Leveque and Prud'homme, three good mathematicians, reviſed the proof ſheets, as did alſo the editor M. Jombert three ſeveral times. M. Didot ſenr. the printer formed the models of the types and founded them on purpoſe, and the editor avers that, during the courſe of the impreſſion, none of the figures came out of their place; a precious advantage which he imputes to the juſtneſs of the principles that M. Didot has eſtabliſhed in his foundery.

THERE is an additional improvement, which I am ſurpriſed none of the editors of our common logarithms has thought of making. What I allude to is the uniting, to the tables of the logarithms of the natural numbers and of the ſines and coſines, the logarithms of their reciprocals (their arithmetical complements *, as they are called). By this means, all the common operations by logarithms might be performed by addition only, without any trouble. The logarithms of the natural numbers might be diſpoſed on the left hand, and thoſe of their reciprocals on the right hand pages. The characteriſtics of the latter, being equal to the difference between 10 and the number of integral figures in the natural numbers, would be as eaſily found as thoſe of the former. The logarithms of the reciprocals of the ſines and coſines might, in each page, be put in the ſame line with the logarithms of the ſines and coſines, having

* The arithmetical complements of the logarithms were firſt thought of by John Speedell, who, in his *new logarithms* firſt publiſhed in 1619, and ſeveral times afterwards, avoided the inconvenience of the ſigns in Napiers logarithms by that contrivance.

having their common differences between them, as the logarithms of the tangents and cotangents, which are reciprocals of each other, have theirs. It is very likely that the prefent edition of the *Tables portatives* will foon be exhaufted. If, in a fecond edition, M. Jombert adopts the propofed amelioration, he will do an effential fervice to the community. 1. The computation might be accomplifhed, by a good arithmetician, in little more than three hours labour every day for half a year. 2. The type and length of the page being the fame, the book would be little more than a fourth part thicker, and would ftill be of a convenient fize.

In the month of May, 1784, there were publifhed propofals for publifhing, by fubfcription, *A Table of Logarithmic fines and tangents, taken at fight to every fecond of the quadrant, accurately computed to feven places of figures befides the index*: *to which will be prefixed a table of the logarithms of numbers from* 1 *to* 100000, *infcribed, by permiffion, to the right honourable and honourable the Commiffioners of the Board of Longitude, by Michael Taylor, one of the computers of the Nautical Ephemeris, and author of a Sexagefimal Table, publifhed by order of the Commiffioners of the Board of Longitude.* The plan of this work was fubmitted to the Board of Longitude, who came to a refolution to give Mr Taylor a gratuity of three hundred pounds fterling towards defraying the expence of printing and publifhing it. This circumftance ought to be a fufficient recommendation of Mr Taylor, and it is to be hoped, that his laborious and ufeful undertaking will meet with the encouragement and recompence from the public which it fo juftly deferves. In the fpecimen annexed to the propofals, the degrees being as ufual at the top and bottom of the page, the feconds occupy the ciples of common Algebra independently of any curve. He was the first also, if I mistake not, that gave the general series for computing the numbers corresponding to given logarithms *. The analytical theory of logarithms, in the Appendix, is nearly on Halley's plan, but was materially finished before the author saw his treatise.

To describe, or enumerate, all the tables of logarithms, which have been published since the invention of these numbers, would be tedious and useless, and indeed next to impossible. We shall restrict ourselves to those which are the most considerable and the most useful.

In the year 1624, Benjamin Ursinus, mathematician to the Elector of Brandenburg, published at Cologne, with his *Trigonometria*, a Table of Napier's logarithms of the sines to every ten seconds of the quadrant. He seems to have been at much pains in computing it, and, in order to obtain the logarithms true to the nearest unit in the eight figure, he supposed the radius followed by an additional zero, as Napier had advised †.

In the same year, Kepler published, at Marpurg, his *Chilias Logarithmorum ad todidem numeros rotundos* &c. and, in the year following, a supplement to it. In this table, the logarithms are of the same kind with those of Napier, but adapted to *sines* in arithmetical progression.

Small tables of the same species of logarithms have been published by T. Simson in his fluxions, by Dr Hutton in his Math. Tab. and by a great many others, to eight places. In Euler's *Introductio in analysin*

X *infinitorum*

* Phil. Transf. for 1695. † Kepl. Epist.

infinitorum, there is a ſmall table of the firſt ten natural numbers with their logarithms to twenty ſix places; and, in Bertrand's work formerly mentioned, there are the logarithms of a great many of the firſt hundred natural numbers, and of ſeveral others, to the ſame number of places. Some of theſe differ from the truth, by ſome units only, in the laſt figure, and the logarithm of 61 is wrong in the ſixteenth figure from the left hand. In the Appendix there is a table of Napier's logarithms of the firſt hundred and one natural numbers to twenty ſeven places.

In the year 1624, Briggs publiſhed at London his *Arithmetica Logarithmica*. This work contains Briggs' or the common logarithms, and their differences, of all the natural numbers from 1 to 20000, and from 90000 to 100000 to fifteen places, including the index or characteriſtic. In *ſome* copies, of which there is one in the Library of the Univerſity of Edinburgh, there is added the logarithms of the numbers from 100000 to 101000, which Briggs had computed after the former had been printed off. Before his death, which happened in 1630, this author completed alſo a table of the logarithmic ſines and tangents to fifteen places, for the hundredth part of every degree of the quadrant, and joined with it the natural ſines, tangents, and ſecants, which he had before calculated. This work which Briggs had committed to the care of Henry Gellibrand, at that time profeſſor of aſtronomy in Greſham College, was tranſmitted to Gouda, where it was printed under the inſpection of Ulacq, and was publiſhed at London in 1633, with the title of *Trigonometria Britannica*.

These tables of Briggs' have not been equalled, for their extenſiveneſs and accuracy together; thoſe of his logarithms that have been re-

examined

examined having feldom been found to differ from the truth by more than a few units in the fifteenth figure.

In the year 1628, Adrian Ulacq of Gouda, in Holland, after filling up the gap betwixt 20000 and 90000, which Briggs had left, republifhed the *Arithmetica Logarithmica*, together with a table of the logarithmic fines, tangents, and fecants, to every minute of the quadrant. Some years afterwards, he publifhed his *Trigonometria Artificialis*, containing Briggs' logarithms of the firft twenty thoufand natural numbers, and the logarithmic fines and tangents, with their differences for every ten feconds of the quadrant. In both thefe works, the logarithms are carried to the eleventh place including the index, and are held in much eftimation for their correctnefs.

Abraham Sharp, of Yorkfhire, publifhed with his *Geometry Improved*, in 1717, a table containing Briggs' logarithms of the firft hundred natural numbers, and of all the prime numbers from 100, to 1100 and of all the numbers from 999980 to 1000020, to fixty two places including the characteriftic. There is the greateft probability of all thefe logarithms being correct. The laft forty-one [from 999980 to 1000020] were verified afterwards by Gardiner.

Tables of the logarithms, carried to fo great a number of places as thofe of Sharp, Briggs, and Ulacq, are feldom ufed; the logarithms to eight places inclufive of the characteriftic being fufficient for all common purpofes. The moft ufeful tables are thofe which have the logarithms correct to the neareft unit in the eight figure, difpofed fo as to take

take up little room, and, at the same time, to afford the easiest and most speedy means of finding the intermediate logarithms, or numbers corresponding to given numbers or logarithms The form of the tables best adapted to answer these purposes was first introduced by Nathaniel Roe, a clergyman in Suffolk, in his *Tabulæ Logarithmicæ*, printed at London in 1633. This form was improved by John Newton, in his *Trigonometria Britannica* published at London in 1658, and by Sherwin in his Mathematical Tables, of which the first edition was printed in 1705. It has received additional improvements in Mr Callet's edition of Gardiner's Tables printed at Paris in 1783.*

The dispositon of the tables is as follows: Each page of the logarithms of the natural numbers is divided into twelve columns. The first column, titled N at top and bottom, contains the natural number. In the second column, marked O, are the logarithms, without the characteristic, of these numbers: the three first figures, belonging to the logarithms of more numbers than one, are separated by a point from the other four figures of the logarithm of the *first* of these numbers and are left out before the other four figures of the logarithms of the rest. In each line of the next nine columns, marked with the nine significant digits 1, 2, 3, &c. are four figures, which, united to the first three isolated figures of the second column in the same line with them, or above them,

* Tables portatives de Logarithmes, publices a Londres par Gardiner, Angmentee et perfectionees dans leur disposition par M. Callet, et corrigees avec la plus scrupuleuse exactitude: contenant les logarithmes des nombre depuis 1 jusqu'a 102960, les logarithmes des sinus and tangentes, de seconde en seconde pour les deux premiers degres et de 10 en 10 secondes pour tous les degres du quart de circle; precedees d'un precis elementaire sur l'explication et l'usage des logarithmes et sur leur application aux calculs d'interets, a la Geometric-pratique, a l'Astronomie et a la Navigation; suivies de plusieurs tables interessantes et d'un discours qui en facilite l'usage. A Paris 1783.

the firſt column: the minutes are diſpoſed along the tops and bottoms of the other columns: immediately below the minutes at top ſtand the characteriſtics, and below *them* the three next common figures of the logarithms; the other four figures filling the columns. It is to be regretted, that an improvement, ſimilar to M. Callet's, has not been adopted in this work, the printing of which was begun before the date of the propoſals.

The tables of logarithms which, with thoſe that have been mentioned, are moſt in eſtimation, are thoſe of the edition of Sherwin, which was corrected and publiſhed by Gardiner in the ſame year (1742) with his own tables—Thoſe by Deparcieux *, and thoſe of the ſmall editions of Ulacq publiſhed at Lyons in 1670, and 1760 †.

The London edition of Gardiner, which has been deſervedly eſteemed as containing the moſt accurate ſet of tables, is not entirely free from errors. There is, at the end of Dr Hutton's tables, a liſt of about fifty errors in the logarithms of the natural numbers, ſines and tangents; twenty of which he himſelf diſcovered in collating the proofs of his book with the like parts of Gardiner's; all of theſe, however, that gentleman obſerves, are not in all the copies of this edition. In the Avignon edition of Gardiner (1770), the errors pointed out by Dr Hutton are above ſeventy. All the errors of the London edition are corrected in the *Tables portatives*, excepting that of the logarithm of the natural numbers 64445.

 Before

* Montucla. † Hutton.

Before concluding this ſection, we ſhall ſay a few words of the logarithms called logiſtic. The logiſtic logarithm of a number of ſeconds is the exceſs of the logarithm of 3600″ above the logarithm of that number of ſeconds. A table of theſe logarithms was firſt given by Strut in his *Aſtronomia Carolina* publiſhed in 1661 *. A ſimilar one is given in ſeveral of the common logarithmic tables.

SECTION

* Tab. portatives.

SECTION VII.

THE USE OF THE LOGARITHMS.

THE general uſe of the logarithms, as was before obſerved, is to convert every ſpecies of multiplication and diviſion into addition and ſubtraction, and to raiſe quantities to any given power, and to extract their roots by eaſy multiplications and diviſions. Examples of theſe operations, particularly in trigonometry, are prefixed to almoſt all the moſt conſiderable tables of logarithms. We beg leave to refer the reader to Gardiner, Callet, Sherwin, and Hutton, where he will find the theory, conſtruction, and application of theſe numbers.

THE theory of the logarithms has put it in our power to ſolve, with great eaſe, an equation in algebra, which before could not be ſolved but with difficulty and tatonnement. In the equation $a^x = b$, if b is the unknown quantity, its value is found by multiplying a by itſelf as often as there are units in $x-1$: Again, if a is the unknown quantity, its value may be found by extracting the xth root of b. But if x is the unknown quantity, algebra, without the logarithms, can furniſh no direct rule for finding its value. This, however, is eaſily accompliſhed by by the affiftance of the logarithms. Let L denote the logarithm of the quantity to which it is prefixed. Now fince $a^x = b$, it is evident that $La = {}^{x}Lb$: but $La^x = xLa$: therefore $xLa = Lb$: therefore $x = \frac{Lb}{La}$.

The Solution of Equations of the form $a^x = b$ is of great importance in political arithmetic. Suppofe that a quantity at firft *m*, being increafed at the end of every equal portion of time by a quantity *c*, augments at the rate *r*; and that it is found, at the end of a number *x* of thefe portions of time, to be augmented to *n*; the equation expreffing the relation of thefe quantities to each other is $(1+r)^x = \frac{n + \frac{c}{r}}{m + \frac{c}{r}}$.

By the help of the logarithms, this formula, among other purpofes, ferves for finding with facility in what time a fum of money *n* might be paid off by finking at firft a fum *m*, and at the end of every year another fum *c*, leaving their intereft *r* to accumulate. In what time, for example, might the national debt of Great Britain, 270 millions of pounds Sterling, be extinguifhed by finking one million every year and allowing its intereft, five per cent per annum, to accumulate? The calculation is as follows.

$$n = 270\,000\,000 \qquad n + \frac{c}{r} = 290\,000\,000 \quad \text{Log.} = 8.4623980$$

$$m = 1\,000\,000 \qquad m + \frac{c}{r} = 21\,000\,000 \quad \text{Log.} = 7.3222193$$

$$c = 1\,000\,000 \qquad \text{Log.} \left(\frac{n + \frac{c}{r}}{m + \frac{c}{r}}\right) = 1.1401787$$

$r =$

$$r=\frac{5}{100}=\frac{1}{20}$$

$$1+r=1+\frac{1}{20}=\frac{21}{20} \quad \text{Log. } 21=1\,.\,3222193$$

$$\frac{c}{r}=20\,000\,000 \qquad \text{Log. } 20=1\,.\,3010300$$

$$\text{Log. } (1+r) \quad =0\,.\,0211893$$

$$x=\frac{1\,.\,1401787}{0\,.\,0211893}=\frac{11401787}{211893}$$

$$\text{Log. } 11401787=7\,.\,0569729$$
$$\text{Log. } \quad 211893=5\,.\,3261167$$

$$\text{Log. } x=1\,.\,7308562,\ x=53\,.\,809$$

In leſs than fifty four years, therefore, the Britiſh nation might get quit of their debt, if they could raiſe annually a million Sterling, over and above the amount of the intereſt of that debt and the expences of government.

The ſame equation under the form

$$n=(m+\frac{c}{r})\times(1+r)^x-\frac{c}{r}$$

ſerves for computing the number n of inhabitants of a country which, having at firſt m inhabitants, has received every year for x years a number c of foreigners, and has increaſed annually at the rate r. For example, ſuppoſe the number of the inhabitants of the United States of North America to be at preſent three millions, that they receive ten thouſand emigrants yearly, and that the population in that country increaſes at the rate of one to twenty per annum; What will be the number of inhabitants of those States a hundred years hence? The calculation is as follows:

$$m = 3000\,000 \qquad m + \frac{c}{r} = 3200\,000 \text{ Log.} = 6.5051500$$

$$c = 10\,000 \qquad 100 \text{ Log. } (1+r) = \text{Log.}(1+r)^{100} = 2.1189300$$

$$r = \frac{1}{20} \qquad \text{Log. } \left(m + \frac{c}{r}\right)(1+r)^{x} = 8.6240800$$

$$\frac{c}{r} = 200\,000 \qquad \left(m + \frac{c}{r}\right)(1+r)^{x} = 420800000$$

$$x = 100 \qquad \frac{c}{r} = 200000$$

$$n = 420600000$$

Hence it appears, that were the lands of the United States extensive enough, and were the same circumstances, favourable to population as at present, to continue for one hundred years, the number of their inhabitants would amount to more than four hundred and twenty millions, which is a good deal greater than twice the number of inhabitants computed to be in all Europe.

The logarithms also, after the invention of fluxions, give rise to a species of calculus called the exponential. This calculus was invented by John Bernoulli and first published in the year 1697 *. It is founded on these two principles: 1. The logarithm of the power of a quantity is equal to the product of its exponent by the logarithm of its root, or $x\text{La} = \text{La}^{x}$: 2. The fluxion of the logarithm of a quantity is proportional to the quotient of the fluxion of that quantity by that quantity

* De Serie, Infin. Jacobi Bernoulli.

city or $L\dot{a} = \frac{\dot{a}}{a}$. The exponential calculus is neceſſary for the inveſtigation of curves, the exponents of whoſe abſciſſes and ordinates, or their functions in the equations to theſe curves, are themſelves variable quantities, *u*, *v*, *z*, &c. Exponential curves, ſuch, for example, as have for the value of their ordinates x^{u}, $x^{u^{v}}$, $x^{u^{v^{z}}}$, &c. are ſaid to be of the firſt, ſecond, third, &c. order. What are the ſubtangents, curvatures, areas, &c. of curves of this nature? Let SMM′, (Fig. XII.) be any curve, its abſcis $CSP' = x$ and ordinate $P'M' = y$ and let there be another curve $\sigma\mu\mu'$ having the ſame abſciſs with the former, and its ordinate $P'M'\mu = z = x^{y}$ for example, let $p'\pi'\nu'$ be an ordinate infinitely near to $p'\mu'$ and $\mu'\pi'$ perpendicular to it, and let $\tau'\mu'$ be a tangent at the point μ': the ſimilar triangles $\tau'p'\mu'$ and $\mu'\pi'\nu'$ give $\tau'p' : p'\mu' :: \mu'\pi' : \pi'\nu'$, therefore the ſubtangent $\tau'p' = \frac{z\dot{x}}{\dot{z}}$: but $z = x^{y}$ therefore $Lz = Lx^{y} = yLx$ therefore $\dot{L}z = \dot{\overline{yLx}}$ that is $\frac{\dot{z}}{z} = y\dot{L}x + \dot{y}Lx = \frac{y\dot{x}}{x} + \dot{y}Lx$, and therefore $\tau'p' = \frac{x\dot{x}}{y\dot{x} + x\dot{y}\,Lx}$. Hence it is evident that the relation of x to y, that is, the Equation to the curve SMM′ being given, the fluxion of y may be expreſſed by ſome function of x, and *its* fluxion may be obtained; which value of the fluxion of y being ſubſtituted in the fraction $\frac{x\dot{x}}{y\dot{x} + x\dot{y}\;Lx}$ and the fluxion of x expunged from its numerator and denominator, there will be obtained a finite expreſſion of the ſubtangent $\tau'p'$ of the curve

curve $\sigma\mu\mu'$. For example, let the curve SMM′ be the logarithmic: we have $y = Lx$ *: therefore $\dot{y} = \frac{\dot{x}}{x}$; therefore $\tau'p' = \frac{x}{2Lx}$. From the value of the subtangent and from the equation ($z = x^{Lx}$ to the curve $\sigma\mu\mu'$ a great many of its properties are easily deduced. The ordinate $S\sigma$ at the summit of the curve is equal to the abciss CS: for $y = Lx = 0$ and $z = x^0 = CS$. The tangent at the point σ is parallel to the axis CSD: for Lx $= 0$ and $\tau'p' = \frac{x^0}{20} = \frac{1}{0}0$. The ordinate cs is an asymptote to the curve $\mu\sigma m$: for $x = 0$ and $Lx = -00$ and therefore $\tau'p' = \frac{0}{200} = 0$. The tangent passing through the point c meets the curve $\sigma\mu\mu'$ at the extremity of the ordinate $z = \sqrt{x}$: for $x = \tau'p' = \frac{x}{2Lx}$; therefore $Lx = \frac{1}{2}$. The tangents to the points M and μ, where $y = \frac{1}{\sqrt{2}}$ and $z = x^{\frac{1}{\sqrt{2}}}$, meet in the same point τ in the axis: For the subtangent of the logarithmic c is $= xLx = \tau p = \frac{x}{2Lx}$; therefore $L^2x = \frac{1}{2}$ and $Lx = \frac{1}{\sqrt{2}}$. The curve $\sigma\mu\mu'$ may be called the *Numerico-Logarithmic*: and if the equation were $(Lx)^x = z$ or $y^x = z$ there would be generated a curve which might be called the *Logarithmo-numeric*.

The above small specimen may suffice for giving an idea of the use of the exponential calculus. The reader will have observed that we have made use of Napier's, or, as they have been called, the natural logarithms. It would have been an easy affair to have made use of any other

* See Appendix.

other logarithms. It may here be obſerved that the logarithmic itſelf, is an exponential curve of the firſt degree or order: for the abſciſs x is of the form c^y, c being a conſtant quantity greater than unity and having 1 for its logarithm.

Those, who wiſh to enter fully into this ſubject, may conſult the Works of John Bernoulli, and the *Analyſe des Infinimens petits* of the Marquis de l'Hopital with M. Varignon's *Eclairciſſements.*

Another uſe of the logarithms is to ſolve the problems of ſailing according to the true chart, independant of a table of meridional parts. It was firſt publiſhed, by Mr H. Bond, about the year 1645, that *the meridian line was analogous to a ſcale of logarithmic tangents of half the complements of the longitudes* *. Nicolas Mercator ſeems to have been the firſt to demonſtrate this property of the meridional line. But he kept his demonſtration ſecret. James Gregory firſt publiſhed a demonſtration of it in his *Exercitationes Geometricæ.* Halley, afterwards, (about the year 1695) gave a much better one in the philoſophical tranſactions. On this ſubject the reader may conſult Robertſon's Navigation, where he will find it treated in a plain manner and illuſtrated with examples.

The logarithms alſo exhibit the aſſymptotic areas of the hyperbola †.

They are likewiſe of great ſervice for the ſummation of infinite ſerieſes in the calculus of fluents. This is true particularly of Napier's

logarithms.

* Phil. Trans. N° 219.

† See Sect. vi. and Appendix.

logarithms. The ſum, for example, of about ſeven hundred millions of terms of the infinite ſeries $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$, &c. is equal to 0.69314718, Napier's logarithm of the number 2.

SECTION

SECTION VIII.

NAPIER'S IMPROVEMENTS IN THE THEORY OF TRIGONOMETRY.

WE obferved before that the Arabs, fetting afide the chords of the double arcs, which rendered Trigonometry very complicated among the ancients, made ufe of the halves of thefe chords to which they gave the name of the *Sinus*. To that ingenious people we owe alfo the three theorems which are the foundation of our modern fpherical trigonometry. By thefe theorems all the cafes of rectangular fpherical triangles and all the cafes of oblique fpherical triangles may be refolved, excepting when the three fides, or the three angles only, are the data. It was Regiomontanus who firft invented two theorems for the folution of thefe two cafes: by which means the theory of trigonometry was perfected. One of thefe theorems which ferves for finding an angle from the three fides is, *The rectangle under the fines of the two fides of any fpherical triangle is to the fquare of the radius; as the difference of the verfed fines of the bafe and the difference of the two fides is to the verfed fine of the vertical angle.* The other theorem, of itfelf, is not fufficient for the purpofe of finding a fide from the three angles.

THIS

This laſt caſe, however, may be reſolved into the former by means of the ſupplemental triangle, ſo called becauſe its ſides are the ſupplements of the angles of the other. This invention is due to Bartholomus Pitiſcus *, who flouriſhed in the beginning of the ſeventeenth century.

The improvements made by Napier on this ſubject are chiefly three. 1. The general rule for the ſolution of all the caſes of rectangular ſpherical triangles, and of all the caſes of oblique ſpherical triangles, excepting the two formerly mentioned. 2. A fundamental theorem by which the ſegments of the baſe, formed by a perpendicular drawn from the vertical angle, may be found, the three ſides being given. This, with the foregoing and the property of the ſupplemental triangle, ſerves for the ſolution of all the caſes of ſpherical triangles. 3. Two proportions for finding by one operation *both* the extremes, the three middle of five contiguous parts of a ſpherical triangle being given.

These theorems are announced by Napier in terms to the following import:

1. Of the circular parts of a rectangular or quadrantal ſpherical triangle. *The rectangle under the radius and the ſine of the middle part is equal to the rectangle under the tangents of the adjacent parts and to the rectangle under the coſines of the oppoſite parts.* The right angle or quadrant ſide being neglected, the two ſides and the complements of the other three natural parts are called the circular parts; as they follow each other as it

* Pitiſco aliquid tribuo in μετάθεσιν arcuum in angulos, et viciſſim. Kep. Epiſt. 295.

it were in a circular order. Of theſe any one being fixed upon as the middle part, thoſe next to it are the adjacent, and thoſe fartheſt from it, the oppoſite parts.

2. *The rectangle under the tangents of half the ſum and half the difference of the ſegments formed at the baſe by a perpendicular drawn to it from the vertical angle of any ſpherical triangle, is equal to the rectangle under the tangents of half the ſum and half the difference of the two ſides.*

3. *The ſines of half the ſum and half the difference of the angles at the baſe of any ſpherical triangle are proportional to the tangents of the half baſe and half the difference of the ſides.*

4. *The coſines of half the ſum and half the difference of the angles of the baſe of any ſpherical triangle, are proportional to the tangents of half the baſe and half the ſum of the ſides.*

NAPIER gives alſo the two following theorems for finding an angle, the three ſides of any ſpherical triangle being given.

5. *The rectangle under the ſines of the two ſides is to the rectangle under the ſines of half the ſum and half the difference of the baſe and the difference of the two ſides, as the ſquare of the radius is to the ſquare of the ſine of half the vertical angle.*

6. *The rectangle under the ſines of the two ſides is to the rectangle under the ſines of half the ſum and half difference of the ſum of the two ſides and the baſe, as the ſquare of the radius is to the ſquare of the coſine of the vertical angle.*

For the demonſtration of the various caſes of the firſt of theſe ſix propoſitions, he refers to the elementary books on trigonometry then in uſe. This propoſition is not ſo ſuſceptible of a direct demonſtration. The demonſtration perhaps the neareſt to a direct one is given in the appendix; of which demonſtration the hint is taken from Napier.

His demonſtration of the ſecond propoſition is extremely elegant and of an uncommon caſt. The reader on theſe accounts, it is preſumed, will be very glad to ſee the ſubſtance of it; which is as follows:

Let a plane MN (Fig. XIII.) touch the ſphere ADP at the point A, the extremity of its diameter PA. Upon the ſurface of the ſphere let there be deſcribed the triangle $A\lambda\gamma$ acute in γ, or $A\lambda\beta$ obtuſe in β. Let the ſine $A\lambda$ and the baſe $A\gamma$ or $A\beta$ be produced to the point P. With the pole λ and diſtance $\lambda\gamma$ or its equal $\lambda\beta$ let the ſmall circle of the ſphere $\beta\gamma\varepsilon\delta$ interſecting λP in ε and λA in δ be deſcribed: and from λ let the arc $\lambda\mu$ be drawn perpendicular to $A\beta\gamma$. $A\gamma$ is the ſum of the ſegments of the baſe and $A\beta$ their difference. $A\varepsilon$ is the ſum of the ſides and $A\delta$ their difference. Let there be ſuppoſed a luminous point in P: The ſhadows, A, b, and c, of the points A, β and γ, upon the plane MN, are in the ſame ſtraight line, becauſe the points A, β, γ and P are in the ſame circular plane: alſo the ſhadow A, d and e, of A, δ and ε, upon the plane MN, are in the ſame ſtraight line, becauſe A, δ, ε and P are in the ſame circular plane. Since PA is perpendicular to the plane MN, the plane triangles PAc, PAb, PAe and PAd are rectangular in A: therefore, to the radius PA, the ſtraight lines Ac, Ab, Ae and Ad, are the tangents of the angles APc or $AP\gamma$, APb or $AP\beta$,

APe

APe or Apε and APd or APδ refpectively. But thefe angles, being at the circumference of the fphere, have for their meafures the halves of the arcs intercepted by their fides: therefore Ac, Ab, Ae and Ad are the tangents of the halves of Aγ, Aβ, Aε and Aδ refpectively. Now (by optics) the fhadow of any circle, defcribed on the furface of the fphere, produced by rays from a luminous point fituated in any point of that furface excepting the circumference of the circle, forms a circle on the plane perpendicular to the diameter at whofe extremity the luminous point is placed: therefore the points c, b, e and d are in the circumference of a circle: therefore $Ac \times Ab = Ae \times Ad$. Q. E. D.

THE third and fourth propofitions are not demonftrated by Napier. He probably deduced them from the fecond in a manner fimilar to that in the appendix; where the reader will find all of thefe and fome other theorems of the fame kind, demonftrated. Napier had left the third propofition under a clumfy form. It was put into the form above given by Briggs in his *Lucubrationes* annexed to the *Canonis Mirifici Conftructio.* This circumftance is not the fole mark of this work being a pofthumous publication.

THE fifth propofition is deduced by Napier from the theorem of Regiomontanus, and it is likely he derived the fixth from the fame fource. To thefe two theorems the logarithms are much more applicable than to that of Regiomontanus.

SINCE Napier's time the chief improvement made in the theory of trigonometry is the application of the calculus of fluxions to it; for which we are indebted to Cotes.

M.

M. Pingre, in the *Memoires de mathematique et de physique* for the year 1756, reduces the solution of all the cases of spherical triangles to four analogies. These four analogies are in fact, under another form, Napier's Rule of the circular parts and his second or fundamental theorem, with its application to the supplemental triangle. Although it would be no difficult matter to get by heart the four analogies of M. Pingre, yet there are few blessed with a memory capable of retaining them for any considerable time. For this reason, the rule for the circular parts, ought to be kept under its present form. If the reader attends to the circumstance of the second letters of the words *tangents* and *cosines* being the same with the first of the words *adjacent* and *opposite*, he will find it almost impossible to forget the rule. And the rule for the solution of the two cases of spherical triangles, for which the former of itself is insufficient, may be thus expressed: *Of the circular parts of an oblique spherical triangle, the rectangle under the tangents of half the sum and half the difference of the segments at the middle part* (formed by a perpendicular drawn from an angle to the opposite side), *is equal to the rectangle under the tangents of half the sum and half the difference of the opposite parts.* By the circular parts of an oblique spherical triangle are meant its three sides and the *supplements* of its three angles. Any of these six being assumed as a middle part, the opposite parts are those two of the same denomination with it, that is, if the middle part is one of the sides, the opposite parts are the other two, and, if the middle part is the supplement of one of the angles, the opposite parts are the supplement of the other two. Since every plane triangle may be considered as described on the surface of a sphere of an infinite radius, these two rules may be applied to plane triangles, provided the middle part be restricted to a *side*.

Thus

Thus it appears that two ſimple rules ſuffice for the ſolution of all the poſſible caſes of plane and ſpherical triangles. Theſe rules, from their neatneſs and the manner in which they are expreſſed, cannot fail of engraving themſelves deeply on the memory of every one who is a little verſed in trigonometry. It is a circumſtance worthy of notice that a perſon of a very weak memory may carry the whole art of trigonometry in his head.

A P P E N D I X.

I.

ANALYTICAL THEORY OF THE LOGARITHMS.

1. LET the consecutive terms of an infinite geometrical progression differ infinitely little one from another; it is evident that, any determined quantity c greater than unity being the basis of the progression, there will be some term $c^x = m$ any given quantity.

2. THE exponents of the terms of that progression are said to be the *logarithms* of those terms: Thus the symbol L denoting the logarithm of the quantity to which it is prefixed, $Lc^{\pm x} = \pm x$. Hence if $c^x = m$; then $Lm = x$ and $L\frac{1}{m} = -x = -Lm$.

THEOREM I.

3. *The logarithm of a product is equal to the sum of the logarithms of its factors.* For since $Lc^x = x$ and $Lc^z = z$ (2), it follows that $Lc^x + Lc^z = x + z$; but $x + z = Lc^{x+z}$ (2) $= Lc^x \times c^z$: therefore $Lc^x \times c^z = Lc^x + Lc^z$. Hence if $c^x = m$ and $c^z = n$ (1); then $Lmn = Lm + Ln$ and $L\frac{m}{n} = Lm - Ln$.

THEOREM.

THEOREM II.

4. *The logarithm of a power is equal to the product of its exponent by the logarithm of its root.* For, since $Lc^x = x$, it follows that $nLc^x = nx$; but $nx = Lc^{nx}$ (2), therefore $Lc^{nx} = nLc^x$. Hence if $c^x = m$, then $Lm^n = nLm$.

PROBLEM I.

5. *To exhibit the logarithm of a given number.* Since $c^0 = 1$, if d is an infinitely small quantity and μ any finite quantity, it is evident that $c^d = 1 + \frac{d}{\mu}$. Now $Lc^d = d$ (2), therefore $d = L(1 + \frac{d}{\mu})$, therefore $id = iL(1 + \frac{d}{\mu}) = L(1 + \frac{d}{\mu})^i$ (4). Let $(1 + \frac{d}{\mu})^i = 1 + a$; we have $id = i\mu(1+a)^{\frac{1}{i}} - i\mu$: therefore, developing the surd quantity $(1 + \frac{d}{\mu})^{\frac{1}{i}}$, making $i = \infty$, and reducing

$$L(1+a) = \mu(a - \frac{a^2}{2} + \frac{a^3}{3} - \&c) \quad - - - - - - - - - - - \quad X$$

Hence, if a is negative,

$$L(1-a) = -\mu(a + \frac{a^2}{2} + \frac{a^3}{3} + \&c) \quad - - - - - - - - - \quad Y$$

Hence, by subtracting Y from X

$$L(1+a) - L(1-a) = L(\frac{1+a}{1-a}) = 2\mu(a + \frac{a^3}{3} + \frac{a^5}{5} + \&c) \quad - \quad - \quad Z$$

6. The above formulæ are of no use for the calculation of the logarithms if a is supposed an integer. Let therefore m and n be any positive numbers, m being greater than n; and

1*mo.* Let $a = \frac{n}{m}$; then $1 + a = \frac{m+n}{m}$, $1 - a = \frac{m-n}{m}$ and $\frac{1+a}{1-a} = \frac{m+n}{m-n}$, and the formulæ X, Y, and Z become, by substitution, A, B, and C.

$$L(\frac{m+n}{m}) = L(m+n) - Lm = \mu(\frac{n}{m} - \frac{n^2}{2m^2} + \frac{n^3}{3m^3} - \&c) \quad - - - \quad A$$

L

$$L\left(\frac{m-n}{m}\right) = L(m-n) - Lm = -\mu\left(\frac{n}{m} + \frac{n^2}{2m^2} + \frac{n^3}{3m^3} + \&c\right) \quad - \; - \; - \quad B$$

$$L\left(\frac{m+n}{m-n}\right) = L(m+n) - L(m-n) = 2\mu\left(\frac{n}{m} + \frac{n^3}{3m^3} + \frac{n^5}{5m^5} + \&c\right) \quad - \; - \quad C$$

2do. Let $a = \frac{n}{m+n}$; then $1 - a = \frac{m}{m+n}$, and the formula Y becomes D

$$L\left(\frac{m}{m+n}\right) = Lm - L(m+n) = -\mu\left(\frac{n}{m+n} = \frac{n^2}{2(m+n)^2} + \frac{n^3}{3(m+n)^3} + \&c\right) \quad D$$

3tio. Let $a = \frac{m}{m+n}$; then $1 - a = \frac{n}{m+n}$, and the formula Y becomes E

$$L\left(\frac{n}{m+n}\right) = Ln - L(m+n) = -\mu\left(\frac{m}{m+n} + \frac{m^2}{2(m+n)^2} + \frac{m^3}{3(m+n)^3} + \&c\right) \; - \; E$$

4to. Let $a = \frac{n}{2m+n}$; then $\frac{1+a}{1-a} = \frac{m+n}{m}$, and the formula Z becomes F

$$L\frac{m+n}{m} = L(m+n) - Lm = 2\mu\left(\frac{n}{2m+n} + \frac{n^3}{3(2m+n)^3} + \frac{n^5}{5(2m+n)^5} + \&c\right) \quad F$$

5to. Let $a = \frac{n}{2m-n}$; then $\frac{1+a}{1-a} = \frac{m}{m-n}$, and the formula Z becomes G

$$L\left(\frac{m}{m-n}\right) = Lm - L\,(m-n) = 2\mu\left(\frac{n}{2m-n} + \frac{n^3}{3(2m-n)^3} + \frac{n^5}{5(2m-n)^5} + \&c\right) G$$

6to. Let $a = \frac{m-n}{m+n}$; then $\frac{1+a}{1-a} = \frac{m}{n}$ and the formula Z becomes H

$$L\frac{m}{n} = Lm - Ln = 2\mu\left(\left(\frac{m-n}{m+n}\right) + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c\right) \quad - \; - \quad H$$

7mo. Let $\frac{n^2}{m^2}$ be substituted for $\frac{n}{m}$ in the formula B: let this new formula be divided by c; and Let $L(m^2 - n^2)$ or $L(m+n) + L)m - n = \sigma$ and $L(m+n)L(m = -n)\; s$: then shall

$$Lm = \frac{1}{2}\sigma + \frac{1}{2}s : \left(\frac{s}{2n} + \frac{n^3}{12m^3} + \frac{7n^5}{180m^5} + \frac{107n^7}{7560m^7} + \frac{1141n^9}{113400m^9} + \&c\right) \quad - \; - \quad I$$

REMARKS.

7. Of three quantities $m-n$, m and $m+n$, in arithmetical progression, the logarithm of the second, being given the logarithms of the other two may be found by one operation, if the odd and even powers of $\frac{n}{m}$ in the serieses A and B are calculated apart.

E e 8.

8. If n is ſuppoſed equal to unity, and if μ (the modulus of the ſyſtem of logarithms to be afterwards determined), conſiſts of a great number of figures, it will be much more convenient, in calculating by the serieſes A, B, C, D, F, and G, to conſider μ as the numerator of each term than as the multiplier of the ſum of the terms.

9. The firſt ſtep $\frac{2\mu n}{2m \times n}$ of the ſeries F will give the logarithms of all numbers greater than 20000 true to fifteen places, if thoſe of all numbers leſs than 20000 are given, and if $2\mu n$ does not exceed a few units.

10. The firſt ſtep $\frac{\sigma}{2} + \frac{\delta n}{4m}$ of the ſeries I will give the logarithms of all numbers greater than 10000 true to nineteen places, if thoſe of all numbers leſs than 10000 are given, and if n does not exceed a few units.

The reader will eaſily ſee that the logarithm of all numbers below m being known, that of $\frac{m+n}{2}$ and conſequently that of $m+n$ and therefore σ as well as δ will be known.

11. Various methods might be taken to compute with eaſe the logarithms of the lower prime numbers. The logarithms, for example, of about two thirds of the primes under 100 may be obtained with little trouble from a table of the continual halfs of the modulus, n being $=1$. The inſpection of the following table will make this evident.

given

given	fought	m	$m+1$	$m-1$	feries
the logar of.					
1	2	2		1	B
2	3	2	3		A
2 and 3	5	2^2	5	3	C
2 and 3	7	2^3	3^2	7	C
2,3 and 5	17	2^4	17	3×5	C
2 and 3	11	2^5	3×11		A
2	31	2^5		31	B
2,3,5 and 7	13	2^6	5×13	$3^2\times7$	C
2 and 3	43	2^7	3×43		A
2,3,5 and 7	19	2×10	3×7	19	C
2,3,5 and 13	41	$2^2\times10$	41	3×13	C
2,3 and 5	79	$2^3\times10$	3^4	79	C
2,5 and 7	23	$2^4\times10$	7×23		A
2,3 and 5	53	$2^4\times10$		3×53	B
2,5 and 11	20	$2^5\times10$		11×29	B
2,3 and 5	71	$2^6\times10$		$3^2\times71$	B
2,3,5 and 7	61	$2^7\times10$	$3\times7\times61$		A

12. THE value of L(1 + 2) was firft given by Nicolas Mercator, who deduced it from a property of the equilateral hyperbola*. The feries *c* was firft demonftrated by James Gregory†. A feries fome what lefs general than *i* was produced by John Keill in his treatife *de Natura* and *arithmetica logarithmorum*: but I think I have fome where feen it attributed to Newton. Some of the other formulæ I believe are new.

PROBLEM II.

13. *To exhibit the modulus of a fyftem of logarithms.* This is effected by fubftituting c for m, and 1 for n, in the equation II. Its value is as follows:

$$\mu = \frac{1}{2\left(\frac{c-1}{c+1}\right)+\frac{2}{3}\left(\frac{c-1}{c+1}\right)^3+\frac{2}{5}\left(\frac{c-1}{c+1}\right)^5+\&c.}$$

REMARK.

* Logarithmotechnia. † Exer. Geom.

REMARKS.

14. In our common ſyſtem of logarithms, c is equal to 10; which gives the following values of μ and its reciprocal to thirty decimal places.

$$\mu = 0.43429\ 44819\ 03251\ 82765\ 11289\ 18917$$
$$\frac{1}{\mu} = 2.30258\ 50929\ 94045\ 68401\ 69914\ 54684$$

15. The modulus of Napier's ſyſtem is unity: for he ſuppoſed the logarithm of a number differing from unity by a very ſmall quantity d to be equal to the ſum or difference of 1 and d: Hence if ${}^{c}L$ denote the common, or Brigg's, logarithm, and ${}^{n}L$, Napier's logarithm of the ſame number; then

$${}^{c}L = (0.43429\ \&c)\ {}^{n}L; \text{ and } {}^{n}L = (2.30258\ \&c)\ {}^{c}L$$

PROBLEM. III.

16. *To exhibit the number of a given logarithm.* We have ſeen that d being $= \frac{1}{\omega}$ and μ a finite quantity, that $c^{d} = 1 + \frac{d}{\mu}$, (5): we have

therefore $c^{x} = \left(1 + \frac{d}{\mu}\right)^{\frac{x}{d}}$, and conſequently

$$c^{x} = 1 + \frac{x}{m} + \frac{x^2}{1.2\mu^2} + \frac{x^3}{1.2.3\mu^3} + \&c \quad - - - - - - - - - - - - \Phi$$

and if x is negative,

$$c^{-x} = 1 - \frac{x}{\mu} + \frac{x^2}{1.2\mu^2} - \frac{x^3}{1.2.3\mu^3} + \&c \quad - - - - - - - - - - - - \Psi$$

Hence, by dividing Φ by Ψ,

$$\frac{c_x}{c^{-x}} = c^{2x} = \frac{1 + \frac{x}{\mu} + \frac{x^2}{1.2\mu^2} + \frac{x^3}{1.2.3\mu^3} \times \quad \&c}{1 - \frac{x}{\mu} \times \frac{x^2}{1.2\mu^2} - \frac{x^3}{1.2.3\mu^3} \times} \quad - - - - - - - - - - \Omega$$

17. If x is greater than μ the above ſeriefes converge ſo ſlowly that that they are of no uſe for finding the number correſponding to a gi-

ven

given logarithm. Let therefore m and n be two numbers differing little from each other, m being greater than n, and

1 *mo.* Let $x = L(\frac{m}{n}) = Lm - Ln = \delta$. Then $c^x = \frac{m}{n}$ and $c^{-x} = \frac{n}{m}$ and the equations Φ and Ψ give

$$m = n(1 + \frac{\delta}{\mu} + \frac{\delta^2}{1.2\mu^2} + \frac{\delta^3}{1.2.3\mu^3} + \text{\&c}) \quad - \quad - \quad - \quad - \quad \text{M}$$

$$n = m(1 - \frac{\delta}{\mu} + \frac{\delta^2}{1.2\mu^2} - \frac{\delta^3}{1.2.3\mu^3} + \text{\&c}) \quad - \quad - \quad - \quad - \quad \text{N}$$

2*do.* Let $x = L(\frac{m}{n})^{\frac{1}{2}} = \frac{1}{2}L(\frac{m}{n}) = \frac{1}{2}Lm - \frac{1}{2}Ln = \frac{1}{2}\delta$: then $c^{2x} = \frac{m}{n}$ and the equation Ω gives

$$m = n\left(\frac{1 + \frac{\delta}{2\mu} + \frac{\delta^2}{1.2.2^2\mu^2} + \frac{\delta^3}{1.2.3.2^3\mu^3} + \text{\&c}}{1 - \frac{\delta}{2\mu} + \frac{\delta^2}{1.2.2^2\mu^2} - \frac{\delta^3}{1.2.3.2^3\mu^3} + \text{\&c}}\right) \quad \text{P}$$

18. More generally, let there be any number n of numbers $m' \nabla m'' \nabla m''' \nabla m'''' \nabla \cdots\cdots \nabla m^{|n|}$ which, taken consecutively, differ little from each other: and let $Lm^{|p|} - Lm^{|p-1|} = \delta^{|p-1|}$ and $\delta'^p - (n-2)\delta''^p + \frac{(n-2)(n-3)}{1.2}\delta'''^p \cdots\cdots \pm \delta^{|n-1|\,p} = \Delta^{|p|}$ (the quantities $|p|$, $|p-1|$, $|n|$, $n-1|$ &c. inclosed in *lines*, expressing simply some terms of the series 1, 2, 3, 4, 5 &c): we have

$$\frac{m''}{m'} - (n-2)\frac{m'''}{m''} + \frac{(n-2)(n-3)m''''}{1.2\;m'''} \cdots\cdots \pm \frac{m^{|n|}}{m^{|n-1|}} = \frac{\Delta'}{\mu} + \frac{\Delta''}{1.2\mu^2} + \frac{\Delta'''}{1.2.3\mu^3} + \text{\&c} \qquad \text{Q}$$

REMARKS.

19. If the logarithms of the first 20000 natural numbers are given, the two first steps of the series $n(1 + \frac{\delta}{\mu} + \frac{\delta^2}{1.2\mu^2})$ of the series M, or $n(1 - \frac{\delta}{\mu} + \frac{\delta^2}{1.2\mu^2})$ of the series N, or the first step $n(\frac{2\mu+\delta}{2\mu-\delta})$ of the series P, will give the number m or n true to about the fourteenth decimal place.

20. The serieses M and N were first given by Halley, in the Philosophical transactions for the year 1695. He exhibited also a series the same with P, but under an inelegant form; probably owing to his having deduced it from the actual division of M by N.

PROBLEM IV.

21. *To exhibit the number whose logarithm is equal to the modulus.* This is effected by the substitution of μ for x in the formula Φ. It's value is as follows

$$c^{\mu} = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c$$

or taking the sum of thirty fractional terms

$$c^{\mu} = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71353$$

II.

II.

A TABLE OF NAPIER's LOGARITHMS,

OF ALL THE NATURAL NUMBERS FROM 1 to 101 to TWENTY SEVEN PLACES.

Num.	Logarithms.	Num.	Logarithms.
1	0.00000.00000.00000.00000.00000.0	21	3.04452.24377.23422.99650.05979.8
2	0.69314.71805.59945.30941.72321.2	22	3.09104.24533.58315.85347.91757.0
3	1.09861.22886.68109.69139.52452.4	23	3.1354[illegible].42159.29149.69080.67528.3
4	1.38629.436[illegible]11.19[illegible].618[illegible].44642.4	24	3.17805.383[illegible]3.47945.[illegible]1964.69416.0
5	1.60943.79124.3410[illegible].3746[illegible].07593.3	25	3.21887.58248.[illegible]8220.74920.15186.7
6	1.79175.94692.28055.00081.24773.6	26	3.25809.65380.21482.04547.01795.6
7	1.94591.01490.55313.30510.53527.4	27	3.29583.68660.04329.0741[illegible].57357.1
8	2.07[illegible]44.154[illegible]6.79[illegible]35.92[illegible]25.16963.6	28	3.33220.45101.752[illegible]3.92393.[illegible]8169.8
9	2.19722.457[illegible].36219.3827[illegible].04[illegible]24.8	29	3.36729.58299.[illegible]6474.02718.327[illegible].3
10	2.30258.50929.94045.68401.79914.6	30	3.40119.73816.62155.37541.32306.[illegible]
11	2.3[illegible]789.527[illegible]7.98370.54406.19435.8	31	3.433[illegible]8.72044.85146.24592.[illegible]1643.3
12	2.4[illegible]49[illegible].[illegible]6497.[illegible]8000.31022.[illegible]7094.8	32	3.4[illegible]73.59027.9[illegible]726.54703.61606.1
13	2.5[illegible]494.93574.[illegible]536.7[illegible]5.[illegible]4874.4	33	3.49650.75614.60480.23545.[illegible]88[illegible].2
14	2.6[illegible].[illegible]329[illegible].[illegible]52[illegible].6[illegible]452.25848.6	34	3.5263[illegible].05246.16161.38[illegible].[illegible]7657.1
15	2.70[illegible].[illegible]2011.0221[illegible].[illegible]5[illegible].[illegible]045.7	35	3.55534.8[illegible]614.89413.[illegible]975.61120.8
16	2.7725[illegible].[illegible]22.[illegible]1.237[illegible].8[illegible].[illegible]	36	3.58351.89384.5[illegible]11[illegible].[illegible]16[illegible].19547.2
17	2.8[illegible]321.[illegible]4[illegible].5[illegible].1[illegible].[illegible].[illegible].2	37	3.61091.7[illegible]126.44224.4445[illegible].[illegible]556.7
18	2.8[illegible]37.[illegible].[illegible]4.[illegible]22[illegible].[illegible].0	38	3.63758.615[illegible]7.2[illegible]385.7[illegible]2.[illegible]25[illegible]5.5
19	2.9[illegible]443.[illegible].[illegible]46.4[illegible].[illegible]274.3	39	3.[illegible]356.16[illegible]1.2964[illegible].42744.[illegible]73[illegible].8
20	2.99573.2[illegible]735.[illegible]90.[illegible]3[illegible].52235.8	40	3.68[illegible]8[illegible].94541.13[illegible]36.302[illegible].2[illegible]37.0

Num.	Logarithms.
41	3.71357.20667.04307.80386.67633.7
42	3.73766.96182.83368.30591.78301.0
43	3.76120.01156.93562.42347.28425.2
44	3.78418.96339.18261.10289.64078.2
45	3.80666.24897.70319.75739.12498.1
46	3.82864.13964.89095.00022.39849.5
47	3.85014.76017.10058.58582.09506.7
48	3.87120.10109.07890.92906.41737.2
49	3.89182.02981.10626.61021.07054.8
50	3.91202.30054.28146.05861.87507.9
51	3.93182.56327.24325.77164.47798.6
52	3.95124.37185.81427.35488.79516.9
53	3.97029.19135.52121.83414.44691.4
54	3.98898.40465.64274.38360.29678.3
55	4.00733.31852.32470.91866.27029.1
56	4.02535.16907.35149.23335.70491.1
57	4.04305.12678.34550.15140.42726.7
58	4.06044.30105.46419.33660.05041.6
59	4.07753.74439.05719.45061.60503.8
60	4.09434.45622.22100.68483.04688.1
61	4.11087.38641.73311.24875.13891.1
62	4.12713.43850.45091.55534.63964.5
63	4.14313.47263.91532.68789.58432.2
64	4.15888.30833.59671.85650.33927.3
65	4.17438.72698.95637.11065.42467.8
66	4.18965.47420.26425.54487.44209.4
67	4.20469.26193.90966.05967.00720.0
68	4.21950.77051.76106.69908.39988.6
69	4.23410.65045.97259.38220.19980.7
70	4.24849.52420.49358.98912.33442.0
71	4.26267.98770.41315.42132.94545.3
72	4.27666.61190.16055.31104.21868.4
73	4.29045.94411.48391.12909.21088.6
74	4.30406.50932.04169.75378.53278.0
75	4.31748.81135.36310.44059.67639.1
76	4.33073.33402.86331.07884.34916.3
77	4.34380.54218.53683.84916.72963.2
78	4.35670.88266.89591.73086.59648.0
79	4.36944.78524.67021.49417.29455.4
80	4.38202.66346.73881.61226.96878.2
81	4.39444.91546.72438.76558.09809.5
82	4.40671.92472.64253.11328.39955.0
83	4.41884.06077.96597.92347.54722.3
84	4.43081.67988.43313.61543.50622.2
85	4.44265.12564.90316.45485.02939.5
86	4.45434.72962.53507.73289.00746.4
87	4.46590.81186.54583.71857.85172.7
88	4.47733.68144.78206.47231.36399.4
89	4.48863.63697.32139.83831.78155.4
90	4.49980.96703.30265.06680.84819.3
91	4.51085.95065.16850.04115.88401.9
92	4.52178.85770.49040.30964.12170.7
93	4.53259.94931.53255.03732.44095.6
94	4.54329.47822.70003.89623.81827.9
95	4.55387.68916.00540.83460.97867.7
96	4.56434.81914.67836.23848.14058.4
97	4.57471.09785.03382.82211.67216.2
98	4.58496.74786.70571.91962.79376.1
99	4.59511.98501.34589.92685.24340.5
100	4.60517.01859.88091.36803.59829.1
101	4.61512.05168.41259.45088.41982.7

III.

TRIGONOMETRICAL THEOREMS.

(1) Lemma 1. The product of the radius by the difference of the verfed fines of two arcs is equal to twice the product of the fines of half the fum and half the difference of thofe arcs.

R (fin V, a—fin V, b) $= 2$ fin $\frac{a+b}{2} \times$ fin $\frac{a-b}{2}$.

(2) Corollary. The product of the radius by the verfed fine of an arc is equal to twice the fquare of the fine of half that arc.

R fin V, $a = 2$ fin^2 $\frac{1}{2}a$.

(3) Lem. 2. The fum of the cofines of two arcs is to their difference as the cotangent of half the fum of thofe arcs is to the tangent of half their difference.

Cof $a +$ cof b : cof a—cof b : : cot $\frac{b+a}{2}$: tang $\frac{b-a}{2}$.

(4) Lem. 3. The fum of the fines of two arcs is to their difference as the tangent of half the fum of thofe arcs is to the tangent of half their their difference.

Sin $a +$ fin b : fin a—fin b : : tang $\frac{a+b}{2}$: tang $\frac{a-b}{2}$.

(5) Lem. 4. The fum of the cotangents of two arcs is to their difference as the fine of the fum of thofe arcs is to the fine of their difference

Cot $a +$ cot b : cot a—cot b : : fin$(b+a)$: fin $(b-a)$.

(6) Lem. 5. The product of the ſine of the ſum of two arcs and the tangent of half that ſum, is to the product of the ſine of their difference and the tangent of half that difference, as the ſquare of the ſine of half their ſum is to the ſquare of the ſine of half their difference.

$$\text{Sin } (a+b) \times \text{tang } \tfrac{a+b}{2} : \text{ſin } (a-b) \times \text{tang } \tfrac{a+b}{2} :: \text{ſin}^2 \tfrac{a+b}{2} : \text{ſin}^2 \tfrac{a \times b}{2}.$$

(7) Lem. 6. The product of the ſine of the ſum of two arcs and the tangent of half their difference, is to the product of the ſine of their difference and the tangent of half their ſum, as the ſquare of the coſine of half their ſum is to the ſquare of the coſine of half their difference.

$$\text{Sin } (a+b) \times \text{tang } \tfrac{a-b}{2} : \text{ſin } (a-b) \times \text{tang } \tfrac{a+b}{2} :: \text{coſ}^2 \tfrac{a+b}{2} : \text{coſ}^2 \tfrac{a-b}{2}.$$

(8) Lem. 7. In right angled ſpherical triangles the coſine of the hypothenuſe is to the cotangent of one of the oblique angles as the cotangent of the other is to the radius.

(9) Lem. 8. In right angled ſpherical triangles the coſine of the hypothenuſe is to the coſine of one of the ſides as the coſine of the other is to the radius.

(10) Lem. 9. In any ſpherical triangle the product of the ſines of the two ſides is to the ſquare of the radius as the difference of the verſed ſines of the baſe and the difference of the two ſides is to the verſed ſine of the vertical angle, Fig. XIV.

$$\text{Sin AB} \times \text{ſin BC} : \text{R}^2 :: \text{ſin V, AC} - \text{ſin V, (AB} - \text{BC)} : \text{ſin V, B *}.$$

(11) Lem. 10. In any ſpherical triangle the product of the ſines of the two ſides is to the ſquare of the radius, as the difference of the verſed ſines of the ſum of the two ſides and the baſe is to the verſed ſine of the ſupplement of the vertical angle, Fig. XIV.

$$\text{Sin AB} \times \text{ſin BC} : \text{R}^2 :: \text{ſin V, (AB + BC)} - \text{ſin V, AC} : \text{ſin V, ſup. B}.$$

(12)

* This is one of Regiomontanus' propoſitions.

(12) The natural parts of a triangle are its three ſides and its three angles.

(13) The circular parts of a rectangular (or quadrantal) ſpherical triangle are the two natural parts adjoining to the right angle (or quadrant ſide) and the *complements* of the other three.

(14) Any one of theſe five being conſidered as a middle part, the two next to it are called the adjacent parts, and the other two the oppoſite parts: Thus, in the triangle *d*AB (Fig. XV.) rectangular in A, if the complement of the angle *d* is taken as a middle part, the adjacent parts are the ſide *d* A and the complement of the hypothenuſe *db*; and the oppoſite parts the ſide *b* A and the complement of the angle *b*.

(15) Of five great circles of the ſphere AB, BC, CD, DE, and EA (Fig. XV.) let the firſt interſect the ſecond; the ſecond, the third; the third, the fourth; the fourth, the fifth; and the fifth, the firſt; at right angles in the points B, C, D, E and A: there are formed, by the interſections mentioned and by thoſe at the reſpective poles *a*, *b*, *c*, *d* and *e* of theſe great circles, five rectangular triangles *d*A*b*, *b*D*e*, *e*B*c*, *c*E*a* and *a*C*d*: and, if theſe poles are joined by the quadrantal arcs *ab*, *bc*, *cd*, *de* and *ea*, there are formed five quadrantal triangles *adb*, *dbe*, *bec*, *eca*, and *cad*. The circular parts in all theſe triangles are the ſame: the *poſition* of theſe equal circular parts with reſpect to one another in each of theſe triangles is different: therefore

(16) What is true of the circular parts of a rectangular triangle is true of thoſe of a quadrantal; and what is true of one middle part and its adjacent and oppoſite parts is true of the other four middle parts and their adjacent and oppoſite parts.

(17) The circular parts of an oblique ſpherical triangle are its three ſides and the *ſupplements* of its three angles.

(18) Any one of theſe ſix being conſidered as a middle part, the two next to it may be called the adjacent parts; the one facing it, the remote part; and the other two, the oppoſite parts: Thus, in the triangle ABC (Fig. XIV.), if the ſide AC is taken as a middle part, the adjacent parts are the ſupplements of the angles A and C; the oppoſite parts, the ſides AB and BC, and the remote part, the ſupplement of the angle B.

(19) Of ſix great circles of the ſphere let the firſt three, AB, BC, and CA, interſect each other at the poles, B, C and A, of the ſecond three, *ca*, *ab* and *bc*: the interſections, *c*, *a* and *b*, of the latter are the poles of the former: there are formed two triangles ABC and *abc* in which the circular parts are the ſame; the poſition of theſe equal circular parts is different in both: therefore

(20) What is true of one middle part and its adjacent, oppoſite, and remote parts, is true of any other middle part and its adjacent, oppoſite, and remote parts.

(21) If an arc *b*BD*d* paſs through the vertices of theſe two triangles, it will be perpendicular to their baſes CDA and *cda*, and the ſegments at the baſe of the one triangle will be the complements of the ſegments at the vertical angle of the other: that is, $CD = 90° - dba$, $AD = 90° - dbc$, $cd = 90° - ABD$, $ad = 90° - DBC$.

(22) If the radius of the ſphere is ſuppoſed infinite, the ſines and tangents of the ſides of a triangle deſcribed on its ſurface, become the ſides themſelves of a plane triangle. Conſequently all the formulæ of ſpherical trigonometry, where the ſines and tangents only of the ſides enter, are applicable to plane trigonometry. Thoſe, however, in which any

functions

functions of all the three angles and only one ſine or tangent of one ſide enter, muſt be excepted.

(23) Of the circular parts we ſhall denote the middle one by M, the adjacent ones by A and *a*, and the oppoſite ones by O and *o*. If the triangle is oblique, the remote part we ſhall call *m*, and the ſegments at a ſide or angle (21) *S* and *s*.

(24) Theorem 1. Of the circular parts (13) of a rectangular (or quadrantal) ſpherical triangle, the product of the radius and the ſine of the middle part, the product of the tangents of the adjacent parts and the product of the coſines of the oppoſite parts, are equal.

Demonſtration. In the right angled ſpherical triangle *dAb* (Fig. XV.) we have coſ *bd* : cot. A*bd* : : cot A*db* : R (8), and coſ *bd* : coſ A*b* : : coſ A*d* : R (9); therefore R × coſ *bd* = cot A*bd* × cot A*db* = coſ A*b* × coſ A*d*; therefore (16)

R × ſin M = tang A × tang *a* = coſ O × coſ *o*.

(25) Corollary 1. In any ſpherical triangle, the ſines of the ſides are proportional to the ſines of the oppoſite angles. For, in the right angled triangles ADB and CDB (Fig. XIII.), R × ſin BD = ſin AB × ſin A, and R × ſin BD = ſin BC × ſin C; therefore ſin AB : ſin BC : : ſin C : ſin A

(26) Cor. 2. In any ſpherical triangle, the ſines of the ſegments of one of its ſides (produced if neceſſary) are proportional to the cotangents of the angles at the extremities of that ſide. For, in the right angled triangles ADB and CDB, R × ſin AD = cot A × tang BD and R × ſin DC = cot C × tang BD; therefore ſin AD : ſin DC : : cot A : cot C

(27) Cor. 3. In any ſpherical triangle, the coſines of any two ſides are proportional to the coſines of the ſegments of the third ſide. For, in the right angled triangles ADB and CDB, R × coſ AB = coſ AD × coſ

DB, and R$\times$cof BC$=$cof CD$\times$cof DB; therefore cof AB : cof BC :: cof AD : cof DC

(28) Remark 1. This theorem ferves for the folution of all poffible cafes of rectanglar or quadrantal fpherical triangles, and for the folution of all poffible cafes of oblique fpherical triangles (by means of the arc drawn from one of its angles perpendicular on the oppofite fide); excepting when the three angles, or the three fides only, are the data.

(29) Rem. 2. This theorem, by confining the middle part to the two fides, (22) ferves alfo for the folution of all poffible cafes of rectangular plane triangles, and for the folution of all poffible cafes of oblique angled plane triangles (by means of the perpendicular drawn from an angle to to the oppofite fide); excepting when the three fides only are the data.

(30) Rem. 3. Were the complements of the two parts adjoining to the right angle or quadrant fide and the other three natural parts taken as the circular parts, the theorem would be,

$$\text{R}\times\text{cof M}=\text{cot A}\times\text{cot }a=\text{fin O}\times\text{fin }o.$$

But the other is preferable, becaufe it is more eafily remembered. The fecond letter of the word *tangent* is the fame with the firft of *adjacent*. It is the fame of the words *cofine* and *oppofite*. If this is attended to, it is hardly poffible to forget the enunciation of the theorem.

(31) Theorem 2. Of the circular parts (17) of an oblique fpherical triangle, the fquare of the fine of half the middle part, is to the fquare of the radius; as the product of the fines of half the fum and half the difference of the fum of the adjacent parts and the remote part, is to the product of the fines of the adjacent parts.

Dem. For fince (Fig. XIV.) fin V. fupp. B : R^2:: fin V, (AB$+$BC)—fin V, AC: fin AB$\times$fin BC (11), it follows that $\text{fin}^2\frac{1}{2}$ fupp. B : R^2:: fin $\left(\frac{\overline{AB+BC}\pm AC}{2}\right)$

$\left(\frac{\overline{AB+BC}\pm AC}{2}\right)$ × ſin $\left(\frac{\overline{AB+BC}-AC}{2}\right)$ ſin AB × ſin BC (2 and 1); therefore (20)

$\text{Sin}^2 \frac{1}{2} M : R^2 :: \text{ſin} \left(\frac{\overline{A+a}\pm m}{2}\right) \times \text{ſin} \left(\frac{\overline{A+a}-m}{2}\right) : \text{ſin. } A \times \text{ſin } a.$

(32) Theorem 3. Of the circular parts of an oblique ſpherical triangle, The ſquare of the coſine of half the middle part is to the ſquare of the radius; as the product of the ſines of half the ſum and half the difference of the remote part and the difference of the adjacent parts, is to the product of the ſines of the adjacent parts.

Dem. For ſince ſin VB : R^2 : : ſin V, AC—ſin V, (AB—BC) : ſin AB + ſin BC (10), it follows that coſ$^2 \frac{1}{2}$ ſupp. B : R^2 : : ſin $\frac{AC\pm\overline{AB-BC}}{2}$ × ſin $\left(\frac{AC-\overline{AB-BC}}{2}\right)$: AB × ſin BC (2) and (1); therefore (20)

$\text{Coſ}^2 \frac{1}{2} M : R^2 :: \text{ſin}\left(\frac{m\pm\overline{A-a}}{2}\right) \times \text{ſin} \left(\frac{m-\overline{A-a}}{2}\right) : \text{ſin } A \times \text{ſin } a.$

(33) Theorem 4. Of the circular parts of an oblique ſpherical triangle, The ſquare of the tangent of half the middle part is to the ſquare of the radius; as the product of the ſines of half the ſum and half the difference of the ſum of the adjacent parts and the remote part, is to the product of half the ſum and half the difference of the remote part and the difference of the adjacent parts.

That is (by comparing the two preceding theorems)

$\text{Tang}^2 \frac{1}{2} M : R^2 :: \text{ſin}\left(\frac{\overline{A+a}\pm m}{2}\right) \times \text{ſin}\left(\frac{\overline{A+a}-m}{2}\right) : \text{ſin}\left(\frac{m\pm\overline{A-a}}{2}\right) \times \text{ſin}\left(\frac{m-\overline{A-a}}{2}\right)$

(34) Theorem 5. Of the circular parts of an oblique ſpherical triangle, The product of the tangents of half the ſum and half the difference of the ſegments of the middle part is equal to the product of the tangents of half the ſum and half the difference of the oppoſite parts.

Dem. For ſince coſ BA : coſ BC : : coſ DA : coſ DC (27) it follows that coſ BA + coſ BC : coſ BA—coſ BC : : coſ DA + coſ DC : coſ DA—coſ

coſ DC; therefore (3) cot $\left(\frac{BC+BA}{2}\right)$: tang $\left(\frac{BC-BA}{2}\right)$: : cot $\left(\frac{DC+DA}{2}\right)$: tang $\left(\frac{DC-DA}{2}\right)$; therefore tang $\left(\frac{DC+DA}{2}\right)$ × tang $\left(\frac{DC-DA}{2}\right)$: : = tang $\left(\frac{BC+BA}{2}\right)$ × tang $\left(\frac{BC-BA}{2}\right)$; therefore (20 and 21)

$$\text{Tang}\left(\frac{S+}{2}\right) \times \text{tang}\left(\frac{S-s}{2}\right) = \text{tang}\left(\frac{O+o}{2}\right) \times \text{tang}\left(\frac{O-o}{2}\right)$$

(35) Rem. 4. By any of the theorems 2, 3, or 4, being given the three ſides or three angles of a ſpherical triangle, may be found any of its angles or ſides; and, confining the middle part to the ſupplement of an angle, being given the three ſides of a plane triangle, may be found (22)

(36) Rem. 5. By theorem 5, being given the three ſides or three angles of a ſpherical triangle, the ſegment of any of its ſides or angles may be found; and confining the middle part to a ſide, being given the three ſides of a plane triangle, the ſegments of any of its ſides may be found.

(37) Rem. 6. By the firſt theorem, and any one of the other four, may be ſolved all the poſſible caſes of ſpherical and plane triangles. Of theſe four, the laſt is the moſt elegant and the moſt eaſily remembered.

(38) Theorem 6. Of the circular parts of an oblique ſpherical triangle, the tangents of half the ſum and half the difference of the ſegments of the middle part are proportional to the ſines of the ſum and the difference of the adjacent parts.

Dem. For ſince ſin CD : ſin DA : : cot C : cot A (26), it follows that ſin CD + ſin DA : ſin CD — ſin DA : : cot C + cot A : cot C — cot A; therefore tang $\left(\frac{CD+DA}{2}\right)$: tang $\left(\frac{CD-DA}{2}\right)$: : ſin (A + C) : ſin (A — C); therefore (20 and 21)

$$\text{Tan}\left(\frac{S+s}{2}\right) : \text{tang}\left(\frac{S-s}{2}\right) :: \text{ſin}\,(A+a) : \text{ſin}\,(A-a)$$

(39)

(39) Rem. 7. By this theorem, being given two ſides and the included angle, or two angles and the included ſide of any triangle, the ſegments of the angle or ſide may be found.

(40) Theorem 7. Of the circular parts of an oblique ſpherical triangle, The tangents of half the ſum and half the difference of the adjacent parts are proportional to the tangents of half the ſum and half the difference of the oppoſite parts.

Dem. For ſince ſin BC : ſin BA : : ſin A : ſin C (25), it follows that ſin BC + ſin BA : ſin BC—ſin BA : : ſin A + ſin C : ſin A— ſin C, therefore (4) tang $(\frac{BC+BA}{2})$: tang $(\frac{BC-BA}{2})$: : tang $(\frac{A+C}{2})$: tang $(\frac{A-C}{2})$: therefore (20)

$$\text{Tang } (\tfrac{A+a}{2}) : \text{tang } (\tfrac{A-a}{2}) :: \text{tang } (\tfrac{O+o}{2}) : \text{tang } (\tfrac{O-o}{2}).$$

(41) Rem. 8. By this theorem, being given two ſides and the included angle of a plane triangle (22), the other angles may be found.

(42) Theorem 8. Of the circular parts of any ſpherical triangle, The tangents of half the middle part and half the difference of the oppoſite parts are proportional to the ſines of half the ſum and half the difference of the adjacent parts.

Dem. For ſince tang $(\frac{S+s}{2})$ × tang $\frac{S-s}{2}$) = tang $(\frac{O+o}{2})$ × tang $(\frac{O-o}{2})$, (34); and tang $(\frac{S+s}{2})$: tang $(\frac{S-}{2})$: : ſin $(A+a)$: ſin $(A-a)$, (38); and tang $\frac{O+o}{2}$: tang $\frac{O-o}{2}$: : tang $(\frac{A+a}{2})$: tang $(\frac{A-a}{2})$, (40) it follows that tang² $(\frac{S+s}{2})$: tang² $(\frac{O-o}{2})$: : ſin $(A+a)$ × tang $\frac{A+a}{2}$: ſin $(A-a)$ × tang $(\frac{A-a}{2})$; therefore (6)

$$\text{Tang } \tfrac{1}{2} M : \text{tang } (\tfrac{O-o}{2}) :: \text{ſin } (\tfrac{A+a}{2}) : \text{ſin } (\tfrac{A-a}{2}).$$

(43)

(43) Theorem 9. Of the circular parts of an oblique ſpherical triangle, The tangents of half the middle part and half the ſum of the oppoſite parts are proportional to the coſines of half the ſum and half the difference of the adjacent parts.

Dem. For ſince $\tan\left(\frac{S+s}{2}\right)\times\tan\left(\frac{S-s}{2}\right)=\tan\left(\frac{O+o}{2}\right)\times\tan\left(\frac{O-o}{2}\right)$, (34); and $\tan\left(\frac{S+s}{2}\right):\tan\left(\frac{S-s}{2}\right)::\sin(A+a):\sin(A-a)$, (38); and $\tan\frac{O-o}{2}:\tan\frac{O+o}{2}::\tan\frac{A-a}{2}:\tan\frac{A+a}{2}$ (40); it follows, that $\tan^2\frac{S+s}{2}:\tan^2\frac{O+o}{2}::\sin(A+a)\tan\frac{A-a}{2}:\sin(A-a)\tan\left(\frac{A+a}{2}\right)$: therefore (7)

$$\text{Tang}\ \tfrac{1}{2}M:\tan\frac{O+o}{2}::\cos\frac{A+a}{2}:\cos\frac{A-a}{2}.$$

(44) Rem. 9. From theſe two theorems it is evident, that, being given two angles and the included ſide, or two ſides and the included angles of any ſpherical triangle, the other two ſides, or the other two angles may be found; and being given two angles and the included ſide of any plane triangle, the other two ſides may be found by *two* analogies only.

From theſe propoſitions are deduced the following

TRIGONOMETRICAL FORMULÆ.

(45) In any ſpherical triangle ABC, Fig. XIV. we have

$$\text{Sin}\ AB\times\sin BC:R^2::\sin\frac{AC+\overline{AB-BC}}{2}\times\sin\frac{AC-\overline{AB-BC}}{2}:\sin^2\tfrac{1}{2}B\ (32)$$

$$\text{Sin}\ AB\times\sin BC:R^2::\sin\frac{\overline{AB+BC}+AC}{2}\times\sin\frac{\overline{AB+BC}-AC}{2}:\cos^2\tfrac{1}{2}B,\ (31)$$

$$\text{Sin}\ \frac{\overline{AB+BC}+AC}{2}\times\sin\frac{\overline{AB+BC}-AC}{2}:R^2::\sin\frac{AC+\overline{AB-BC}}{2}\times\sin\frac{AC-\overline{AB-BC}}{2}:\tan^2\tfrac{1}{2}B\ (33)$$

$$\text{Sin}\ A\times\sin C:R^2::-\cos\frac{\overline{A+C}+B}{2}\times\cos\frac{\overline{A+C}-B}{2}:\sin^2\tfrac{1}{2}AC$$

$$\text{Sin}\ A\times\sin C:R^2::\cos\frac{B+\overline{A-C}}{2}\times\cos\frac{B-\overline{A-C}}{2}:\cos^2\tfrac{1}{2}AC$$

coſ

$\text{Cos}\,\frac{B+\overline{A-C}}{2}\times\text{cos}\,\frac{B-\overline{A-C}}{2} : R^2 :: -\text{cos}\,\frac{\overline{A+C}+B}{2}\times\text{cos}\,\frac{\overline{A+C}-B}{2} : \text{tang}^2\,\frac{1}{2}AC$

$\text{Tang}\,\frac{1}{2}AC : \text{tang}\,\frac{BC+BA}{2} :: \text{tang}\,\frac{BC-BA}{2} : \text{tang}\,\frac{CD-DA}{2}$

$\text{Sin}\,(A+C) : \text{sin}\,(A-C) :: \text{tang}\,\frac{1}{2}AC : \text{tang}\,\frac{CD-DA}{2}$

$\text{Cot}\,\frac{1}{2}B : \text{tang}\,\frac{A+C}{2} :: \text{tang}\,\frac{A-C}{2} : \text{tang}\,\frac{CDB-DBA}{2}$

$\text{Sin}\,(BC+BA) : \text{sin}\,(BC-BA) :: \text{cot}\,\frac{1}{2}B : \text{tang}\,\frac{CBD-DBA}{2}$

$\text{Tang}\,\frac{BC+BA}{2} : \text{tang}\,\frac{BC-BA}{2} :: \text{tang}\,\frac{A+C}{2} : \text{tang}\,\frac{A-C}{2}$

$\text{Sin}\,\frac{A+C}{2} : \text{sin}\,\frac{A-C}{2} :: \text{tang}\,\frac{1}{2}AC : \text{tang}\,\frac{BC-BA}{2}$

$\text{Cos}\,\frac{A+C}{2} : \text{cos}\,\frac{A-C}{2} :: \text{tang}\,\frac{1}{2}AC : \text{tang}\,\frac{BC+BA}{2}$

$\text{Sin}\,\frac{BC+BA}{2} : \text{sin}\,\frac{BC-BA}{2} :: \text{cot}\,\frac{1}{2}B : \text{tang}\,\frac{A-C}{2}$

$\text{Cos}\,\frac{BC+BA}{2} : \text{cos}\,\frac{BC-BA}{2} :: \text{cot}\,\frac{1}{2}B : \text{tang}\,\frac{A+C}{2}$.

(46) In any plane triangle ABC, Fig. XVI. we have (22)

$AB\times BC : R^2 :: \left(\frac{AC+\overline{AB-BC}}{2}\right)\times\left(\frac{AC-\overline{AB-BC}}{2}\right) : \text{sin}^2\,\frac{1}{2}B$

$AB\times BC : R^2 :: \left(\frac{\overline{AB+BC}+AC}{2}\right)\times\left(\frac{\overline{AB+BC}-AC}{2}\right) : \text{cos}^2\,\frac{1}{2}B$

$(\overline{AB+BC}+AC)\times(\overline{AB+BC}-AC) : R^2 :: (AC+\overline{AB-BC})\times(AC-\overline{AB-BC})$
$: \text{tang}^2\,\frac{1}{2}B$

$AC : BC+BA :: BC-BA : CD-DA$

$\text{Sin}\,(A+C) : \text{sin}\,(A-C) :: AC : CD-DA$

$BC+BA : BC-BA :: \text{tang}\,\frac{A+C}{2} : \text{tang}\,\frac{A-C}{2} :: \text{cot}\,\frac{1}{2}B : \text{tang}\,\frac{A-C}{2}$
$:: \text{cot}\,\frac{1}{2}B : \text{tang}\,\frac{CDB-DBA}{2}$

$\text{Sin}\,\frac{A+C}{2} : \text{sin}\,\frac{A-C}{2} :: AC : BC-BA$

$\text{Cos}\,\frac{A+C}{2} : \text{cos}\,\frac{A-C}{2} :: AC : BC+BA$.

IV.

[illegible]

IV.

THE HYPERBOLA AS CONNECTED WITH THE LOGARITHMS.

1. WHILE a ſtraight line PM (Fig. XVII.) moves parallel to itſelf along the indefinite ſtraight line CPD with a velocity always proportional to the diſtance of its extremity P from a fixed point C, let its other extremity M approach to or recede from P, ſo that PM may deſcribe equal ſpaces in equal times: The point P will deſcribe a part PP′ or Pp' of the ſtraight line CD, while the point M deſcribes a correſponding part MM′ or Mm of the curve m'SM′.

2. If the motion is ſuppoſed to have begun at P, the area PM M′P′ or PM $m'p'$ is the logarithm of the abſciſs CP′ or Cp'.

3. In order that equal ſpaces may be deſcribed in equal times, it is evident that the greater or ſmaller the abſciſs CP′ or Cp' becomes with regard to CP, the ſmaller or greater muſt the ordinate P′M′ or $p'm'$ become with regard to PM; Therefore CP′ : CP : : PM : P′M′, or Cp' : CP : : PM : $p'm'$; Therefore the product of any abſciſs by the correſpondent ordinate is a conſtant quantity: Therefore

4. The curve m'SM′ is a hyperbola having CD for one of its aſſymptotes, and Cs, parallel to the ordinates, for the other.

5. From this manner of conceiving the generation of the hyperbola might be deduced the properties of that curve and of the logarithms. That CD and Cs, for instance, touch the curve at an infinite distance from C appears from this: When the abscifs is infinite, the ordinate must be zero, and when the abscifs is zero, the ordinate must be infinite, in order that their product may equal the finite quantity PM $\times$ CP: And that the logarithm of CP is zero appears from this; PM is length without breadth and therefore no space.

6. Let $CP = a$, $PM = \mu$, $PP' = x$ and $P'M' = y$; we have (3) $y = \frac{a\mu}{a+x}$, or, developing the fraction $\frac{a}{a+x}$ in the manner first taught by Nicolas Mercator*,

$$Y = \mu\left(1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \&c\right)$$

7. It is evident that the space PMM'P' is equal to the sum of all the ordinates $y' + y'' + y''' +$ &c. on the abscifs x. If the abscifs is supposed to be divided into an infinite number of infinitely small and equal parts, the abscissæ corresponding to the ordinates y', y'', y''', &c. may be called 1, 2, 3, &c: therefore (6)

$$y' = \mu\left(1 - \frac{1}{a} + \frac{1}{a^2} - \frac{1}{a^3} + \&c\right)$$
$$y'' = \mu\left(1 - \frac{2}{a} + \frac{2^2}{a^2} - \frac{2^3}{a^3} + \&c\right)$$
$$y''' = \mu\left(1 - \frac{3}{a} + \frac{3^2}{a^2} - \frac{3^3}{a^3} + \&c\right)$$
$$y^x = y = \mu\left(1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \&c\right)$$

therefore

$$y' + y'' + y''' + \&c \ldots + y = \mu \left\{ \begin{array}{l} + \frac{1^0}{a^0} + \frac{2^0}{a^0} + \frac{3^0}{a^0} + \&c \ldots + \frac{x^0}{a^0} \\ - \frac{1^1}{a^1} - \frac{2^1}{a^1} - \frac{3^1}{a^1} - \&c \ldots - \frac{x^1}{a^1} \\ + \frac{1^2}{a^2} + \frac{2^2}{a^2} + \frac{3^2}{a^2} + \&c \ldots + \frac{x^2}{a^2} \\ - \frac{1^3}{a^3} - \frac{2^3}{a^3} - \frac{3^3}{a^3} - \&c \ldots - \frac{x^3}{a^3} \\ + \&c, \&c. \end{array} \right\}$$

Now,

* Logarithmotechnia

Now, as was firſt demonſtrated by Wallis*, the ſum $1^n + 2^n + 3^n +$ &c. continued to infinity, that is to x^n in this caſe, being equal to $\frac{x^{n+1}}{n+1}$; we have

$$PMM'P' = L(a+x) = \mu\left(x - \frac{x^2}{2a} + \frac{x^3}{3a^2} - \&c\right)$$

and, if x is negative,

$$PMm'p' = L(a-x) = -\mu\left(x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \&c\right)$$

8. The quantity μ depends on the angle $DCs = \Phi$ formed by the aſſymtotes and the diſtance $MN = m$ of the point M of the curve from the aſſymptote CD; as is evident from its value $\mu = \frac{rm}{Sin\,\Phi}$, where r denotes the radius of the circle.

* Arith. Infinit.

SECTION.

V.

PROPERTIES OF THE LOGARITHMIC.

1. WHILE two points A and B are in S, (Fig. XVIII.) moving in oppoſite directions along the indefinite ſtraight line CSD with a velocity always proportional to their diſtance from a fixed point C, let all the points in SD and all the points in SC move in oppoſite directions perpendicularly to CSD with any uniform velocity; and in the inſtant that A or B paſſes through any point P′ or *p*′ let the point which left P′ or *p*′ ſtop in M′ or *m*′; A and B will deſcribe the axis, while the points that move perpendicular to it, deſcribe all the ordinates, or the area of the curve *m*′SM′.

2. This curve is called the logarithmic, becauſe its ordinate PM, P′M′ &c. are the logarithms of its abſciſſæ CP, CP′, &c.

3. The ordinate C*s*, at the finite extremity C of the axis, is an aſſymptote to the curve: for, as the point that moves from S towards C cannot arrive at C in any finite time, the point that left C will move on for ever.

4. The ordinate PM, a tangent to the curve at whoſe extremity M meets the point C, is called the modulus of the logarithmic. We ſhall call PM the logarithmic modulus and CS the numeric modulus.

5. Let the portions $P\pi$ and $P'\pi'$ of the axis be ſuppoſed deſcribed in equal times; and let the ſtraight lines $M\nu$, $M\nu'$ be drawn perpendicular to the ordinates $\pi\mu$, $\pi'\mu'$: we have $CP : CP' :: P\pi : P'\pi'$ and $\nu\mu = \nu'\mu'$: But, if the equal times are infinitely ſmall, the arcs $M\mu$ and $M'\mu'$ are ſtraight lines and the right angled triangles CPM and $M\nu\mu$, ſimilar; conſequently $CP : PM :: M\nu$ or $P\pi : \nu\mu$; therefore $CP' : PM :: P'\pi' : \nu\mu$ or $\nu'\mu'$.

6. *To draw a tangent to any point M′ of the Logarithmic.* Upon the ordinate P′M′ take $P'L' = PM$; join the points C and L′ and draw parrallel to CL′ the ſtraight line $M'\tau'$ meeting the axis in the point τ'; $\tau'M'$ touches the curve in the point M′: For ſince (5) $CP' : PM$ or $P'L' :: P'\pi'$ or $M'\nu' : \nu'\mu'$, the triangles CP′L′ and $M'\nu'\mu'$ are ſimilar; therefore $M'\mu'$ is parrallel toCL′; therefore &c: Hence,

7. The ordinates to the aſſymptote, MQ, MQ′, &c. have for their logarithms its abſciſſæ CQ, CQ′: and

8. The ſubtangents CQ, C′Q′, &c. upon the aſſymptote are all equal to the logarithmic modulus PM.

9. The ſubtangent T′P′ upon the axis is to the ordinate P′M′ as the abſciſs CP′ is to the modulus PM; For the triangles T′P′M′ and C′P′M′ are ſimilar: Hence,

10. The ſubtangents upon the axis are to each other as the products of the abſciſſæ and ordinates.

11. The ſubnormal P′N′ upon the axis is to the ordinate as the logarithmic modulus to the abſciſs: For the triangles CP′L′ and M′P′N′ are ſimilar: Hence,

12. The ſubnormals upon the axis are to each other as the quotients of the ordinates and abſciſſæ.

13

13 The ſubtangent is to the ſubnormal as the ſquare of the abſciſs to the ſquare of the logarithmic modulus.

14. Let PM $=\mu$, CP$'=z$ and P$'$M$'=y$: and let $z^1, z^2, z^3, \ldots z^n$ be any number of abſciſſæ in geometrical progreſſion; $S', S'', S''', \ldots S^n$, the correſpondent ſubtangents, and $\sigma', \sigma'', \sigma''' \ldots \sigma^n$, the correſpondent ſubnormals upon the axis:—we have, (10) and (12)

$$n(n+1)(2n+1)y^2 = 6(S'\sigma'+S''\sigma''+S'''\sigma'''+\ldots+S^n\sigma^n)$$

$$\frac{z^2}{\mu^2}\left(\frac{1+z^n}{1+z}\right)\left(\frac{1-z^n}{1-z}\right)=\frac{S'}{\sigma'}+\frac{S''}{\sigma''}+\frac{S'''}{\sigma'''}+\ldots+\frac{S^n}{\sigma}$$

$$y^{2n}=\frac{S'\sigma'}{1^2}\times\frac{S''\sigma''}{2^2}\times\frac{S'''}{3^2}\times\ldots\times\frac{S^n\sigma^n}{n2}$$

$$\left(\frac{z^{n\times 1}}{\mu^2}\right)^n=\frac{S'}{\sigma'}\times\frac{S''}{\sigma''}\times\frac{S'''}{\sigma'''}\times\ldots\times\frac{S^n}{\sigma^n}$$

15. Let the numeric modulus CS $=m$ and SP$'=x$, the denominations of PM and P$'$M$'$ continuing as before: we have P$'\pi=\dot{x}$ and $\nu\mu'=\dot{y}$. Now $m+x : \mu :: \dot{x} : \dot{y}$, (5); therefore $\dot{y}=\frac{\mu\dot{x}}{m+x}$; or if $m=1$ $\dot{y}=\mu\dot{x}(\frac{1}{1+x})=\mu\dot{x}\,(1-x+x^2-x^3+\&c)$ and therefore

$$y=\mu(x-\tfrac{x^2}{2}+\tfrac{x^3}{3}-\&c)=\text{Log. }(1+x).$$

16. Let the area of any portion S$'$M$'$P$'$ of the curve $=$ A: we have $\dot{A}=z\dot{y}$; therefore $A=\int z\dot{y}=zy-\int y\dot{z}$: but $y=\frac{\mu\dot{z}}{z}$ (5); or (15); therefore

$\int y\dot{z}=\int\mu\dot{z}=\mu z$; therefore $A=zy-\mu z+C$: but when $A=o$, then $z=m$ and $y=o$; therefore $o=-\mu m+C$; therefore $C=\mu m$ and $A=z\,(y-h)+\mu m$, that is

17. The area of any portion of the logarithmic is equal to the rectangle under the abſciſs and the difference of the ordinate and the logarithmic

garithmic modulus, together with the rectangle under the moduli: Hence

18. The rectangle CL, under the moduli, is equal to the area SMP contained by the logarithmic modulus PM, the portion of the axis MS, and the arc SM; or to the area $Sm\mu C$ contained by the numeric modulus SC, the assymptote $C\mu$, and the infinite branch mS of the curve.

FINIS.

LIST OF BOOKS,

QUOTED OR CONSULTED, TO ELUCIDATE

THE

LIFE AND WRITINGS,

OF

JOHN NAPIER

OF

MERCHISTON.

ARCHIMEDIS Syrucasani Arenarius, &c. Eutocii Ascalonitæ in hanc comment. cum versione et notis Joh. Wallis.

Dictionaire Historique et Critique par M. Pierre Bayle. A Rotterdam, 1720. Folio passim.

Balcarres' Memoirs.

Bernoulli Ars conjectandi et tractatus de seriebus infinitis. Basileæ, 1713. 4to.

——— Opera Omnia. Lausannæ et Genevæ. 1742.

Biographia Britannica.

Boscovich de Cycloide et Logistica.

Arithmetica Logarithmica sive logarithmorum chiliades triginta; pro numeris naturali serie crescentibus ab unitate ad 20.000 et a 90.000 ad 100.000: quorum ope multa perficiuntur arithmetica problemata et geometrica. Hos numeros primus invenit clarissimus vir Johannes Neperus Baro Merchistonii: eos autem ex ejusdem sententia mutavat, eorumque ortum et usum illustravit Henricus Briggius, in celeberrima Academia Oxoniensi Geometriæ Professor Savilianus. Deus nobis usuram vitæ dedit et ingenii, tanquam pecuniæ nulla præstituta die. Londini 1624. Folio.

Boethius de Arithmetica.

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Christophori Clavii Bambergensis, e Soc. Jesu, Opera Mathematica. Moguntiæ 1611. Folio.

——— de Astrolabio. Vol. III.

Chambers' Dictionary, 2 vols Folio.

Craufurd's Peerage of Scotland.

——— lives of the officers of State.

Crugerus Pref. in Praxin Trig.

Nouveau Dict. Hist. et Crit. pour servir de supplement au Dict. de M. Bayle, par Jaques George de Chauffepie. Amsterdam, 1756. Folio passim.

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Hervarti ab Hohenburgh opera. 1610.

Mathematical Tables containing Common, Hyperbolic and Logistic logarithms; also sines, Tangents

tangents, ſecants and verſed ſines, both natural and logarithmic, &c. to which is prefixed a large and original hiſtory of the diſcoveries and writings relating to thoſe ſubjects &c. By Charles Hutton, LLD. F. R. S. and Prof. Math Royal Acad. Woolwich. Lond. 1785.

Humes Hiſtory of the Stuarts.

Keill de Log.

Joannes Keppleri aliorumque Epiſtolæ mutuæ. Lipſiæ 1718. Folio paſſim.

——— Ephemerides novæ motuum Coeleſtium ab anno 1617.

Leyburnes Recreations. Folio. 1694.

Lilly's Life. London, 1721. 8vo.

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Newtoni Principia. Amſt. 1723.

Phil. Tranſact. London.

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Vitæ quorundam erudiſſimorum et illuſtrium virorum, ſcriptore Thoma Smith. Londini 1707. Commentariolus de vita et ſcriptis D. Henrici Briggii.

Stifellii Arithmetica integra.

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Sir Thomas Urquhart of Cromertie's Triſſotetras. London, 1650. 4to.

—— Tracts. Edinburgh, 1774. 8vo.

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Rabdologiæ ſeu numerationis per virgulas libri duo. Edinburgi, 1617. 12mo.

A plain diſcovery of the whole Revelation of St John, &c. by John Napier of Merchiſton. Edinburgh, 1593. 4to. by Andro Hart.

Mirifici Logarithmorum Canonis deſcriptio &c, Authore Joanne Nepero Barone Merchiſtonii Edinburgæ 1614 apud And. Hart.

Imitatio Nepeirea a Henrico Briggs. MSS. 1614. in the Britiſh Muſeum.

Ayſcongles Cat. of the MSS in the Britiſh Muſeum. London. 2 vols 4to.

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The lives of the Profeſſors of Greſham College by John Ward, Prof. of Rhetoric in Greſham College. F. R. S. London 1740. Folio.

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Trigonometria Britannica. Goudæ 1633. Fol.

Memoires de Mathematique et de Phyſique de l'annee 1756. tirees des regiſtres de l'Academie Royale des Sciences a Amſterdam 1768. La trigonometric ſpherique reduite a quatre analogies par M. Pingre.

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ons by Lowthorp et Motte. 5 vols 4to. passim.

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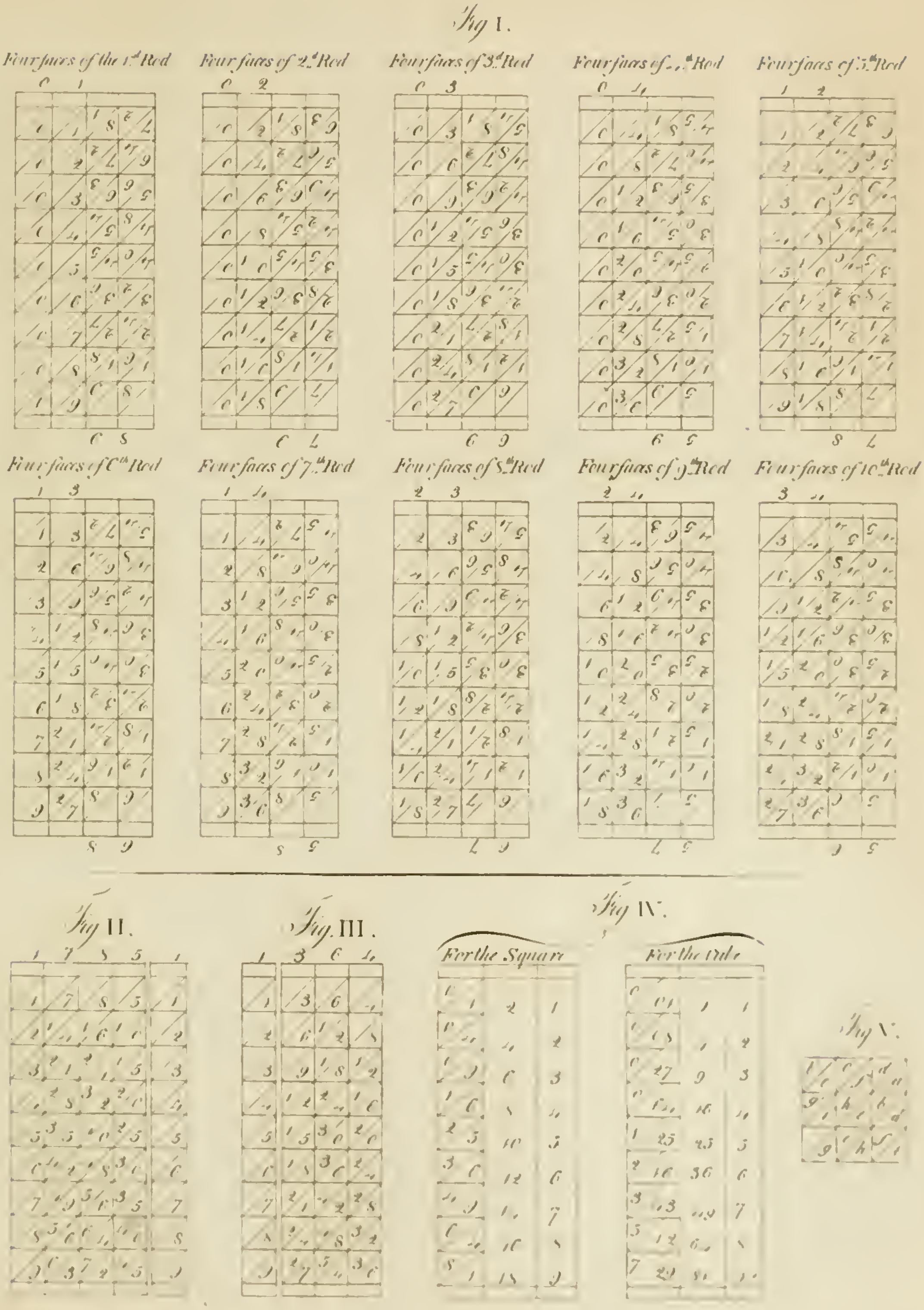
Trig. Plan. et Spher. Elem. item de Natura et Arithmetica logarithmorum tractatus brevis. Oxon. 1723. 8vo.



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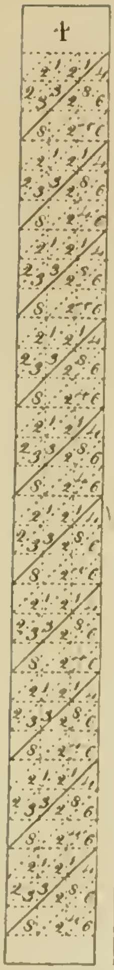


Fig. VI.

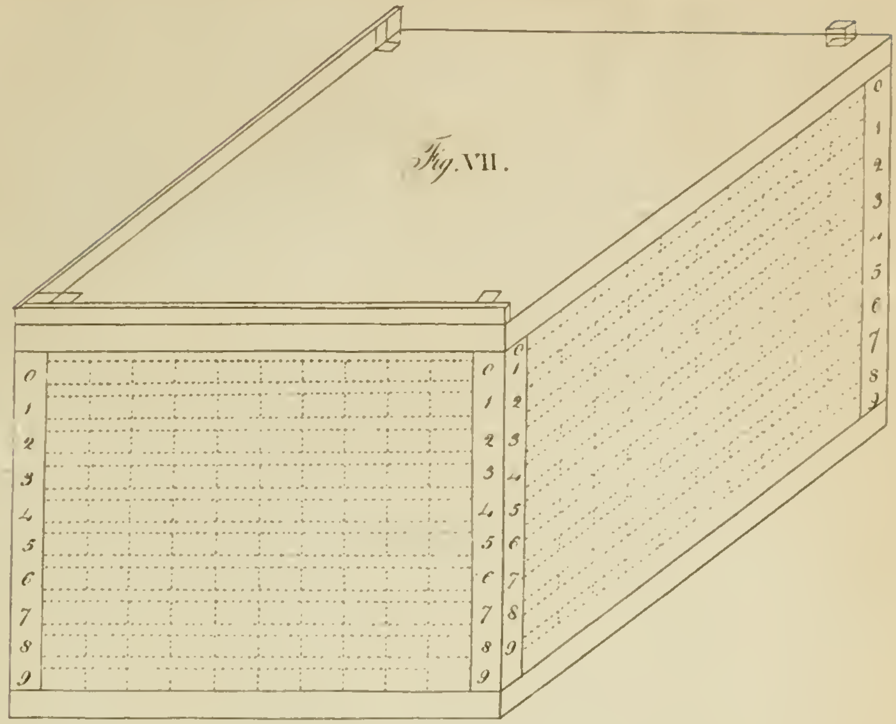
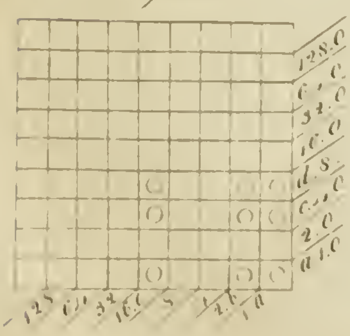


Fig. VII.

Fig. VI.

Fig. VIII.



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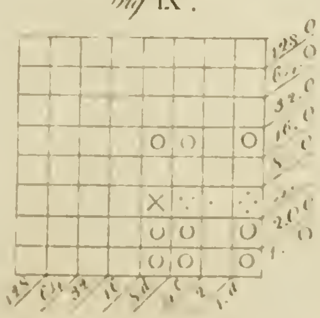
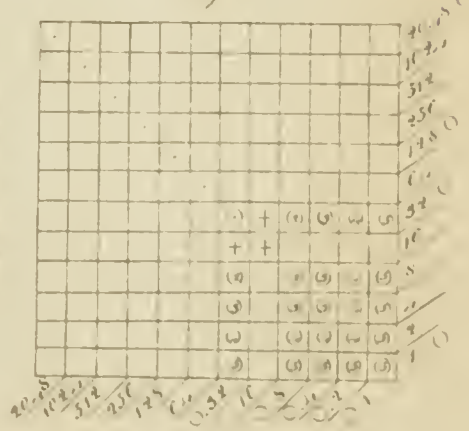


Fig. X.



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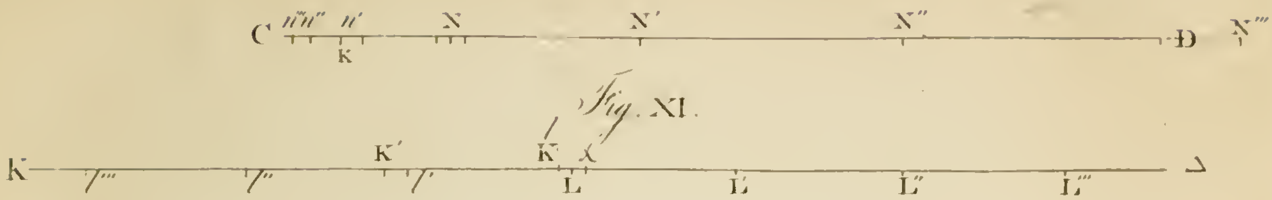


Fig. XI.

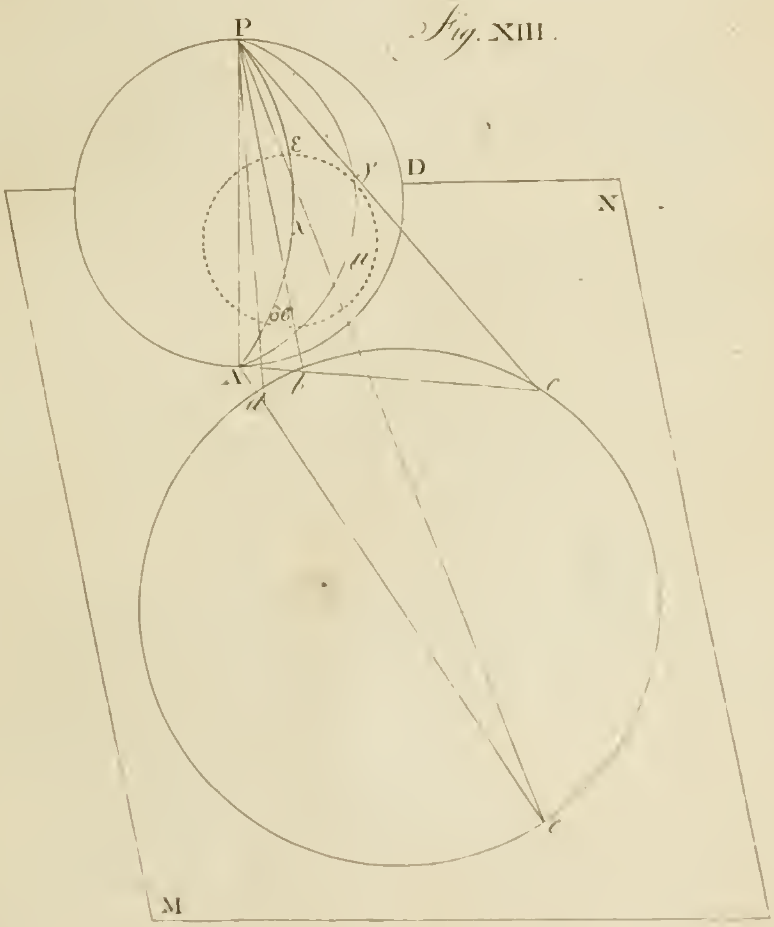


Fig. XIII.

Fig. XII.

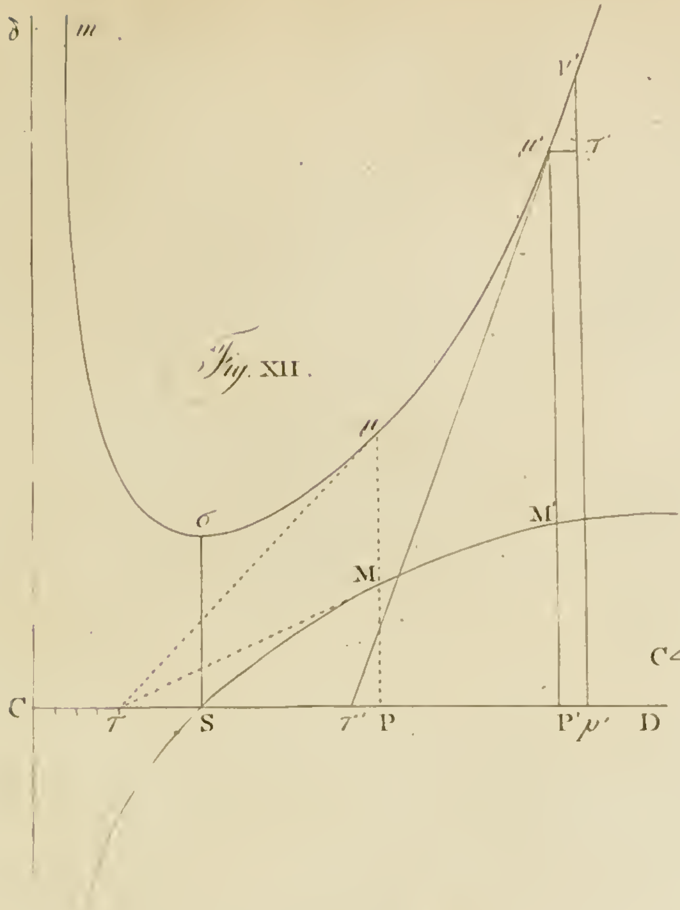


Fig. XVI.

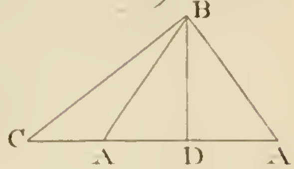


Fig. XIV.

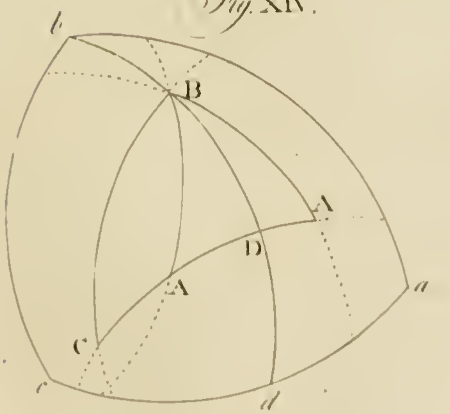
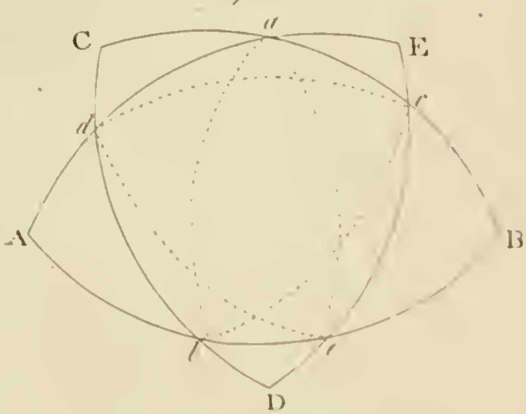


Fig. XV.



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Engraved for the Life of Napier

Plate V

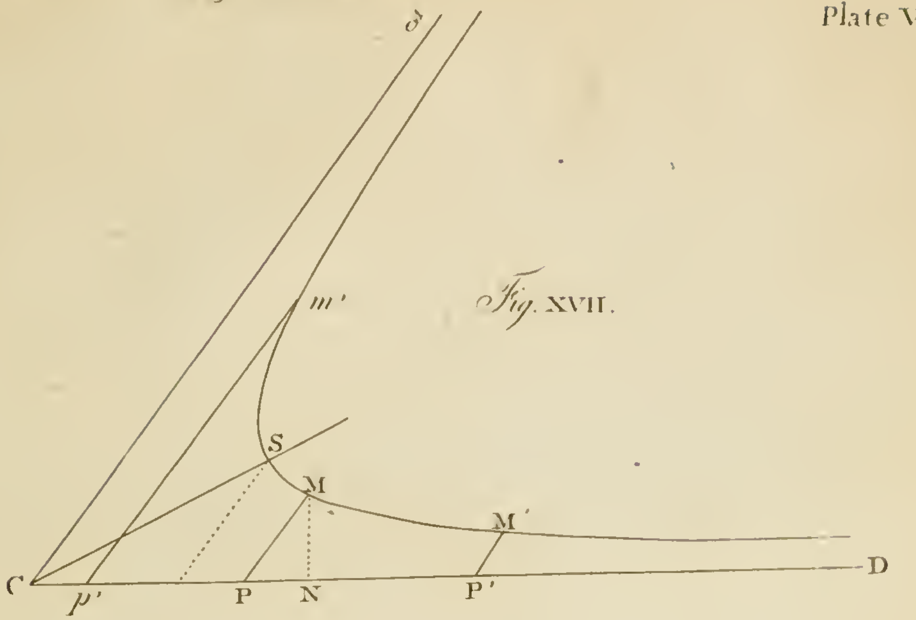

Fig. XVII.

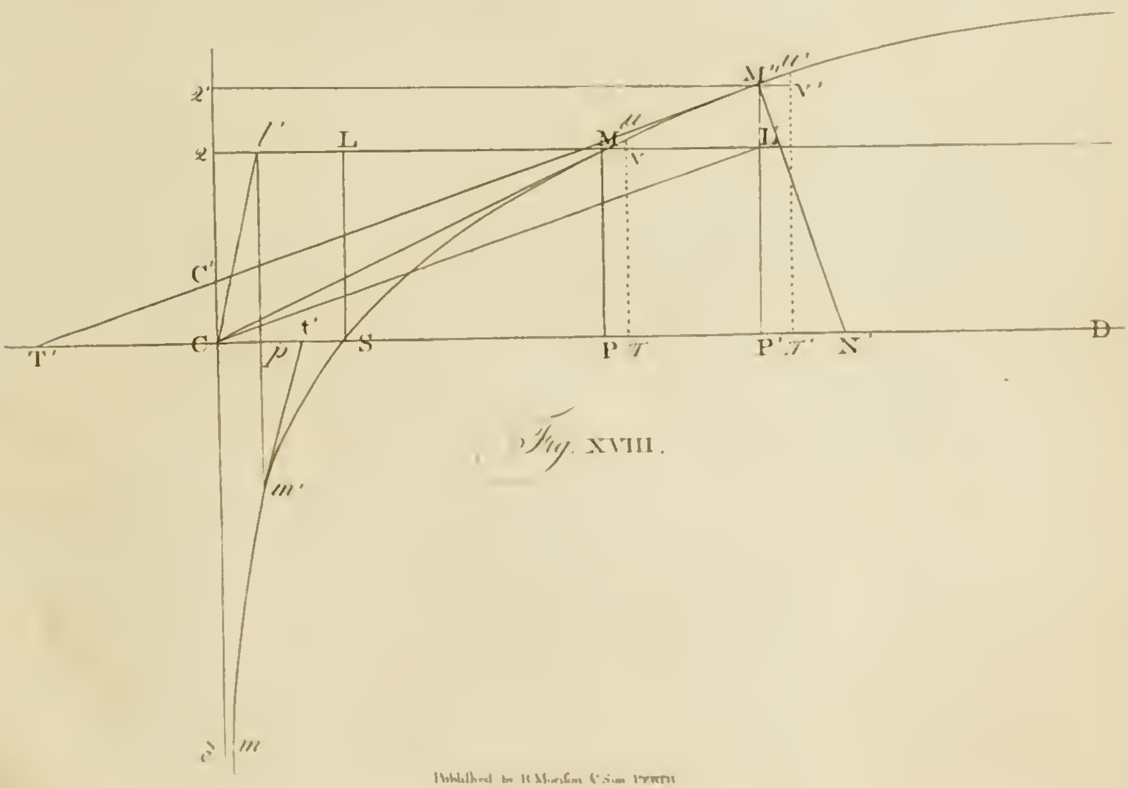

Fig. XVIII.

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